KLEIN-GORDON EQUATION WITH FINITE SQUARE WELL - NUMERICAL SOLUTION

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Reference: W. Greiner: *Relativistic Quantum Mechanics (Wave Equations)*; 3rd Edition, Springer-Verlag (2000); Section 1.11, Example 1.14.

Post date: 11 Jan 2018.

In the last post, we looked at the Klein-Gordon equation with a finite spherical square well. The potential has the form

$$V(r)=\begin{cases} -V_0 & r\le R\\ 0 & r>R \end{cases} \tag{1}$$

We saw that the solution for a general angular momentum number l involved Bessel and Hankel functions, and found that for $l=0$ (the s state), continuity of the wave function and its derivative at $r=R$ gave the condition

$$k_i\cot(k_iR)=ik_o \tag{2}$$

where

$$k_i=\frac{\sqrt{(\varepsilon+V_0)^2-m_0^2c^4}}{\hbar c};\quad \text{for } r\le R \tag{3}$$

$$k_o=\frac{\sqrt{\varepsilon^2-m_0^2c^4}}{\hbar c};\quad \text{for } r>R \tag{4}$$

Numerically solving 2 for the energy ε gives the quantized energy levels. Although Greiner gives a few plots of the energy as a function of the nuclear charge Z, he doesn't actually show how to solve 2, so I thought it would be interesting to use Maple to do this. I'll use Greiner's values, which assume that the potential for $r\le R$ has the constant value

$$V_0=\frac{Ze^2}{R} \tag{5}$$

Greiner actually gives $V_0=-\frac{Ze^2}{R}$, but the minus sign is already present in 1, so V_0 should be positive.

The radius of the well is given by

$$R=r_0A^{1/3} \tag{6}$$

with

In the caption to his first graph, Greiner says that $R = r_0 A^{1/2}$ but this is wrong.

$$r_0 = 1.2 \text{ fm} \tag{7}$$

and A being the number of nucleons (protons and neutrons) in the nucleus, which he takes as

$$A = 2.5Z \tag{8}$$

This is reasonable, as once we get beyond the lightest elements, atomic nuclei contains more neutrons than protons.

For a pion, we have

$$m_0 c^2 = 139.577 \text{ MeV} \tag{9}$$

and for the basic charge we can use

$$e^2 = \alpha \hbar c = \frac{197.32698}{137.036} = 1.44 \text{ MeV} \cdot \text{fm} \tag{10}$$

using $\hbar c = 197.32698 \text{ MeV} \cdot \text{fm}$ and the fine structure constant$\alpha = \frac{1}{137.036}$.

Plugging all this into 2 means we have

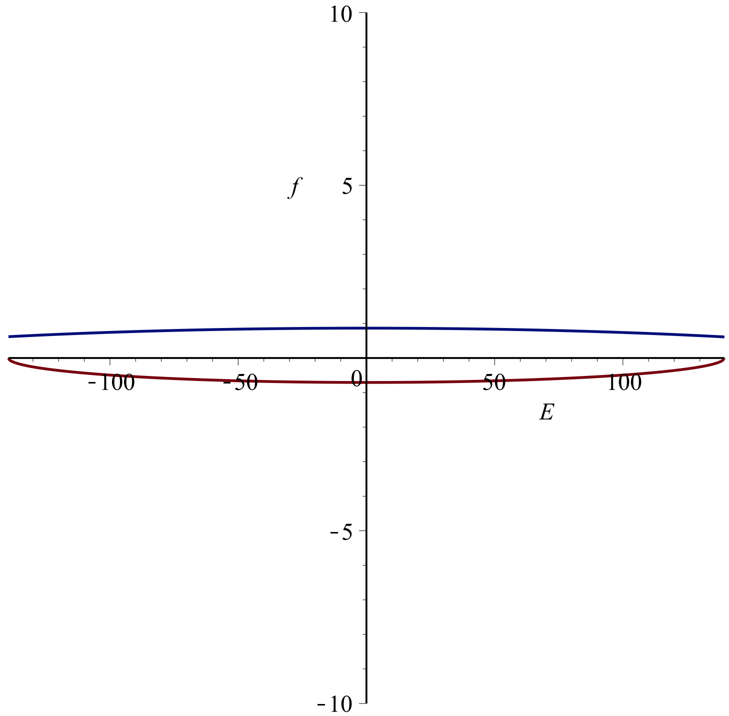
$$V_0 = 1.44 \frac{Z}{r_0 (2.5Z)^{1/3}} = 0.884 Z^{2/3} \text{ MeV} \tag{11}$$

$$k_i = \frac{\sqrt{\left(\varepsilon + 0.884 Z^{2/3}\right)^2 - 19482}}{197.32698} \text{ fm}^{-1} \tag{12}$$

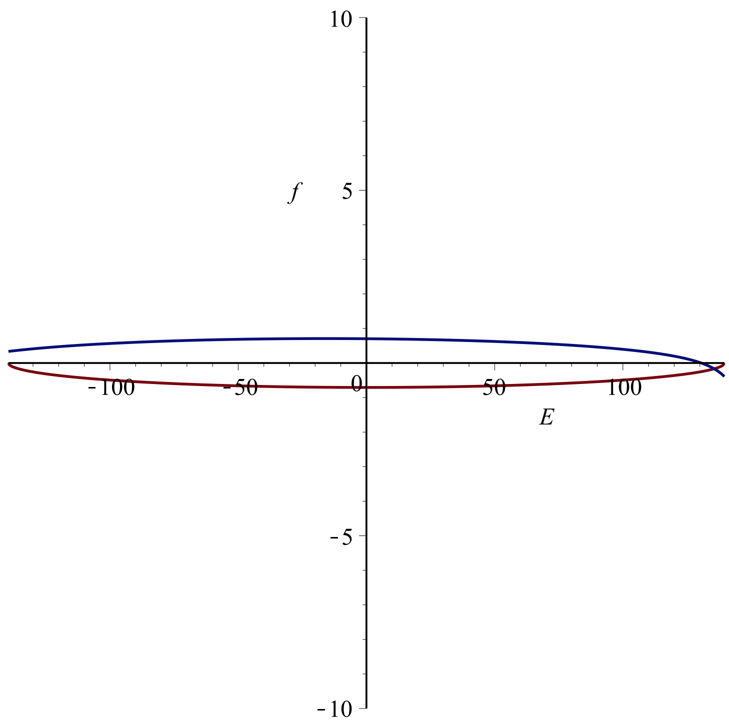
$$k_o = \frac{\sqrt{\varepsilon^2 - 19482}}{197.32698} \text{ fm}^{-1} \tag{13}$$

We can now specify values for Z and find ε from a numerical solution of 2. Before we plunge into the graphs, though, it's a good idea to see what we'd expect from the solution. For small Z, we would expect fewer bound states, since the well depth is shallow. As we increase Z, we'd expect the energy of the lowest bound state to decrease (become further down from $m_0 c^2$), and we might also expect there to be more bound states with higher energies. This is the sort of behaviour we saw for the finite square well in one dimension using the Schrödinger equation.

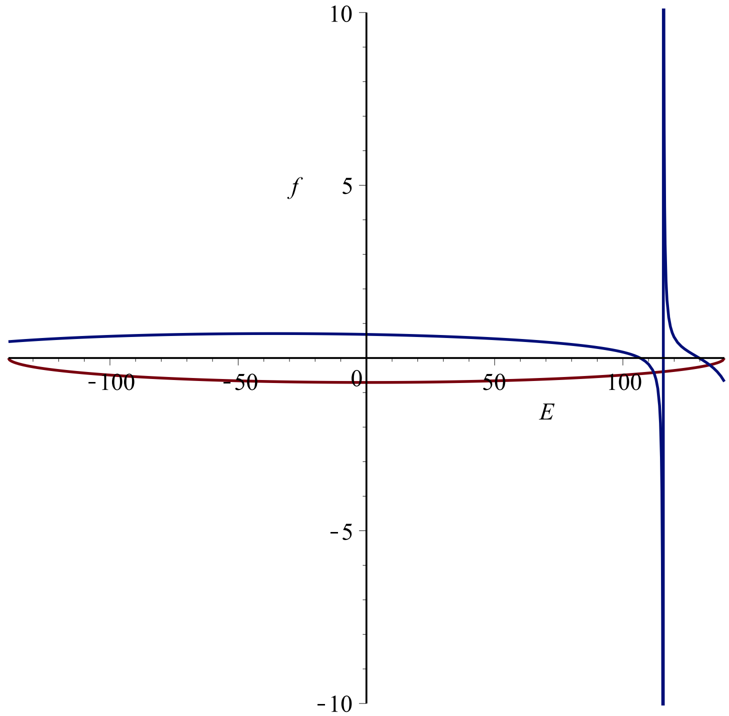
For $Z = 1$, we get the plot (where the energy is on the x axis and ranges between -139.577 MeV and $+139.577$ MeV):



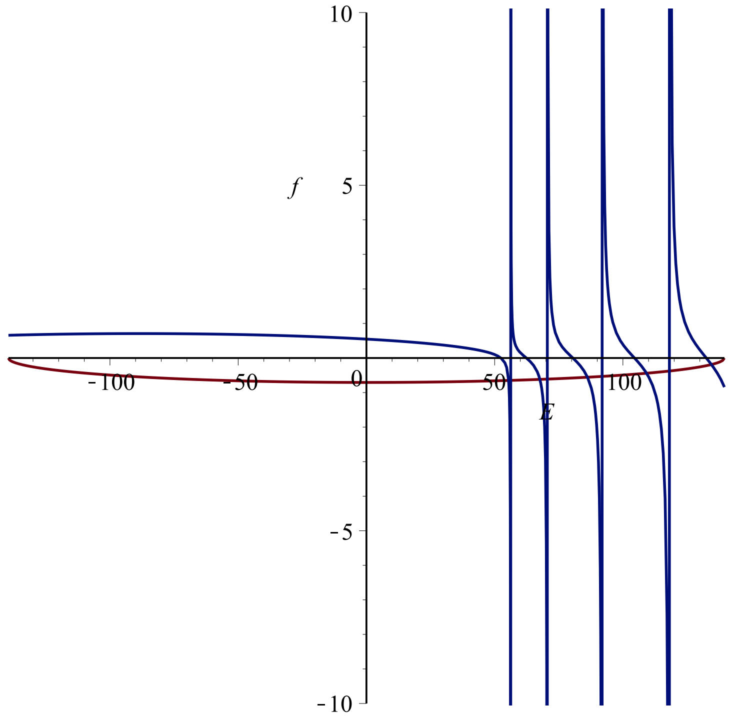
The red curve is k_o (which is the same for all plots since it doesn't depend on Z), and the blue curve is k_i. Since there is no intersection, there is no bound state for $Z = 1$. The smallest value of Z that gives a solution is $Z = 77$, which gives $\varepsilon = 139.384$ MeV and has the graph shown:



If we increase Z further, we find that the cotangent term in 2 hits its first singularity. Beyond this point, another bound state shows up. Here is $Z = 260$:

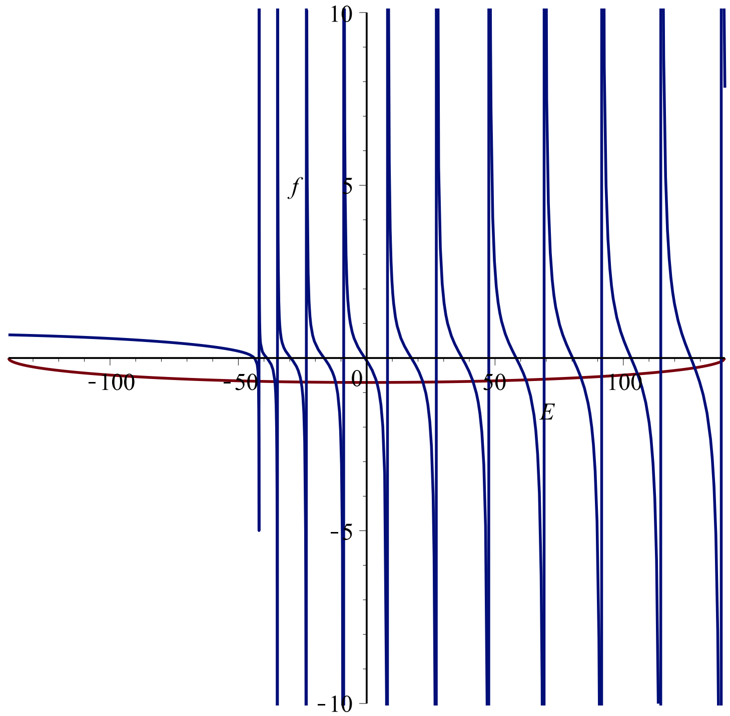


Increasing Z further intoduces more singularities in the cotangent, and more bound states. Here is $Z = 1000$:



We can see that there are now 5 bound states, but for the ground state, we always take the state with lowest energy, here just over 50 MeV.

Finally, if we increase Z enough, the total energy becomes negative, so the binding energy is actually greater than the rest energy of the pion. Here is $Z = 3000$:



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