

KLEIN-GORDON EQUATION WITH SCALAR 1/R POTENTIAL

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Reference: W. Greiner: *Relativistic Quantum Mechanics (Wave Equations)*; 3rd Edition, Springer-Verlag (2000); Section 1.11, Exercise 1.16.

Post date: 16 Jan 2018.

In this exercise, Greiner introduces an unusual form of the Klein-Gordon equation, in which the potential is added to the $m_0^2 c^4$ term rather than modifying the four-momentum term. He uses a scalar interaction of the form

$$W(r) = -\frac{Z\alpha}{r} \quad (1)$$

We've already looked at the K-G equation for a particle in an electromagnetic field, where the interaction was introduced by modifying the four-momentum according to

$$p^\mu \rightarrow p^\mu - \frac{e}{c} A^\mu \quad (2)$$

This is as a result of the Hamiltonian for the electromagnetic force being given by

$$H = \frac{|\mathbf{p} - e\mathbf{A}/c|^2}{2m} + e\phi \quad (3)$$

which, when translated into four-vector notation results in the replacement given by 2. This is known as minimal coupling of the momentum to the four-potential A^μ .

In this case, we are coupling a scalar interaction given in general by $U(r)$ to the square of the mass. We start with the 3-d radial K-G equation that we had when considering the Coulomb potential:

$$\left[-\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{l(l+1)}{r^2} \right] u(r) = \frac{(\epsilon - V)^2 - m_0^2 c^4}{\hbar^2 c^2} u(r) \quad (4)$$

This equation includes the electromagnetic coupling for a Coulomb potential in the $(\epsilon - V)^2$ factor. In our case, we remove this coupling and insert instead a coupling of the scalar interaction to the $m_0^2 c^4$ term. Removing the electromagnetic coupling gives

$$\left[-\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{l(l+1)}{r^2} \right] u(r) = \frac{\varepsilon^2 - m_0^2 c^4}{\hbar^2 c^2} u(r) \quad (5)$$

and coupling to the $m_0^2 c^4$ term gives

$$\left[-\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{l(l+1)}{r^2} \right] u(r) = \frac{\varepsilon^2 - m_0^2 c^4 - U^2(r)}{\hbar^2 c^2} u(r) \quad (6)$$

After the usual substitution

$$u(r) = \frac{R(r)}{r} \quad (7)$$

we get Greiner's equation (2):

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + \frac{\varepsilon^2}{\hbar^2 c^2} - \frac{m_0^2 c^4}{\hbar^2 c^2} - \frac{U^2(r)}{\hbar^2 c^2} \right] R(r) = 0 \quad (8)$$

Greiner now introduces a couple of dimensionless parameters to simplify things. First we have

$$r' \equiv r \frac{\hbar c}{m_0 c^2} \quad (9)$$

Greiner has r' and r the wrong way round in his eqn (3).

Inserting this into 8 we get, after multiplying through by $\hbar^2 c^2 / m_0^2 c^4$:

$$\left[\frac{d^2}{dr'^2} - \frac{l(l+1)}{r'^2} + \frac{\varepsilon^2}{m_0^2 c^4} - 1 - \frac{U^2(r')}{m_0^2 c^4} \right] R(r') = 0 \quad (10)$$

Note that all the terms in the square brackets are now dimensionless.

After another couple of parameters are introduced:

$$b^2 \equiv 1 - \frac{\varepsilon^2}{m_0^2 c^4} \quad (11)$$

$$d \equiv \frac{Z\alpha}{2b} \quad (12)$$

$$\rho \equiv 2br' \quad (13)$$

I use $d = \frac{Z\alpha}{2b}$ rather than Greiner's c in order to avoid confusion with the speed of light.

we transform 10 into

$$\left[\frac{d^2}{d\rho^2} - \frac{l(l+1)}{\rho^2} - \frac{1}{4} + \frac{d}{\rho} \right] R(\rho) = 0 \quad (14)$$

By examining the asymptotic behaviour for $\rho \rightarrow 0$ and $\rho \rightarrow \infty$, Greiner arrives at his eqn (15):

$$R(\rho) = N\rho^{l+1} e^{-\rho/2} F(\rho) \quad (15)$$

The last term in Greiner's eqn (9) should be c/ρ , not ρ/c .

where N is a normalization constant and F is a function to be determined. By inserting this into 14 and calculating the derivatives (I checked this using Maple; the calculation gets a bit messy so I'll just quote the result) we get

$$\rho \frac{d^2 F}{d\rho^2} + (2l + 2 - \rho) \frac{dF}{d\rho} + (d - l - 1)F = 0 \quad (16)$$

In Greiner's eqn (16), there should be a + after the $\frac{dF(\rho)}{d\rho}$ term.

The solution is a confluent hypergeometric function

$$F(\rho) = {}_1F_1(l + 1 - d, 2l + 2, \rho) \quad (17)$$

This function blows up at large ρ at a rate that cannot be compensated by the $e^{-\rho/2}$ in 15 unless its first argument is a negative integer or zero (this condition results from the series form of ${}_1F_1$ which must terminate). We therefore get the quantization condition

$$l + 1 - d = -n_r \quad (18)$$

where n_r is a positive integer. Defining the principal quantum number n as

$$n \equiv l + 1 + n_r \quad (19)$$

we can find the energy from 11 and 12.

$$d = \frac{Z\alpha}{2\sqrt{1 - \frac{\varepsilon^2}{m_0^2 c^4}}} = n \quad (20)$$

which gives the energies

$$\varepsilon = \pm \sqrt{1 - \frac{Z^2 \alpha^2}{4n^2}} m_0 c^2 \quad (21)$$

The energy doesn't depend on the orbital angular momentum number l , but from 19 we see that since n_r must be a positive integer, once we specify the principal quantum number n , then $l + n_r = n - 1$, so $l = 0, 1, \dots, n - 1$ just as in the hydrogen atom from the Schrödinger equation.