# KLEIN-GORDON EQUATION WITH SCALAR 1/R POTENTIAL

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Reference: W. Greiner: *Relativistic Quantum Mechanics (Wave Equations)*; 3rd Edition, Springer-Verlag (2000); Section 1.11, Exercise 1.16.

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In this exercise, Greiner introduces an unusual form of the Klein-Gordon equation, in which the potential is added to the $m_0^2c^4$ term rather than modifying the four-momentum term. He uses a scalar interaction of the form

$$W(r) = -\frac{Z\alpha}{r} \tag{1}$$

We've already looked at the K-G equation for a particle in an electromagnetic field, where the interaction was introduced by modifying the four-momentum according to

$$p^\mu \to p^\mu - \frac{e}{c}A^\mu \tag{2}$$

This is as a result of the Hamiltonian for the electromagnetic force being given by

$$H = \frac{|\mathbf{p} - e\mathbf{A}/c|^2}{2m} + e\phi \tag{3}$$

which, when translated into four-vector notation results in the replacement given by 2. This is known as minimal coupling of the momentum to the four-potential $A^\mu$.

In this case, we are coupling a scalar interaction given in general by $U(r)$ to the square of the mass. We start with the 3-d radial K-G equation that we had when considering the Coulomb potential:

$$\left[-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{l(l+1)}{r^2}\right]u(r) = \frac{(\varepsilon - V)^2 - m_0^2c^4}{\hbar^2c^2}u(r) \tag{4}$$

This equation includes the electromagnetic coupling for a Coulomb potential in the $(\varepsilon - V)^2$ factor. In our case, we remove this coupling and insert instead a coupling of the scalar interaction to the $m_0^2c^4$ term. Removing the electromagnetic coupling gives

$$\left[-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{l(l+1)}{r^2}\right]u(r)=\frac{\varepsilon^2-m_0^2c^4}{\hbar^2c^2}u(r) \tag{5}$$

and coupling to the $m_0^2c^4$ term gives

$$\left[-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{l(l+1)}{r^2}\right]u(r)=\frac{\varepsilon^2-m_0^2c^4-U^2(r)}{\hbar^2c^2}u(r) \tag{6}$$

After the usual substitution

$$u(r)=\frac{R(r)}{r} \tag{7}$$

we get Greiner's equation (2):

$$\left[\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}+\frac{\varepsilon^2}{\hbar^2c^2}-\frac{m_0^2c^4}{\hbar^2c^2}-\frac{U^2(r)}{\hbar^2c^2}\right]R(r)=0 \tag{8}$$

Greiner now introduces a couple of dimensionless parameters to simplify things. First we have

Greiner has $r'$ and $r$ the wrong way round in his eqn (3).

$$r'\equiv r\frac{\hbar c}{m_0c^2} \tag{9}$$

Inserting this into 8 we get, after multiplying through by $\hbar^2c^2/m_0^2c^4$:

$$\left[\frac{d^2}{dr'^2}-\frac{l(l+1)}{r'^2}+\frac{\varepsilon^2}{m_0^2c^4}-1-\frac{U^2(r')}{m_0^2c^4}\right]R(r')=0 \tag{10}$$

Note that all the terms in the square brackets are now dimensionless.

After another couple of parameters are introduced:

I use $d=\frac{Z\alpha}{2b}$ rather than Greiner's $c$ in order to avoid confusion with the speed of light.

$$b^2\equiv 1-\frac{\varepsilon^2}{m_0^2c^4} \tag{11}$$

$$d\equiv\frac{Z\alpha}{2b} \tag{12}$$

$$\rho\equiv 2br' \tag{13}$$

we transform 10 into

The last term in Greiner's eqn (9) should be $c/\rho$, not $\rho/c$.

$$\left[\frac{d^2}{d\rho^2}-\frac{l(l+1)}{\rho^2}-\frac{1}{4}+\frac{d}{\rho}\right]R(\rho)=0 \tag{14}$$

By examining the asymptotic behaviour for $\rho\to 0$ and $\rho\to\infty$, Greiner arrives at his eqn (15):

$$R(\rho)=N\rho^{l+1}e^{-\rho/2}F(\rho) \tag{15}$$

where $N$ is a normalization constant and $F$ is a function to be determined. By inserting this into 14 and calculating the derivatives (I checked this using Maple; the calculation gets a bit messy so I'll just quote the result) we get

$$\rho\frac{d^2F}{d\rho^2}+(2l+2-\rho)\frac{dF}{d\rho}+(d-l-1)F=0 \tag{16}$$

In Greiner's eqn (16), there should be a + after the $\frac{dF(\rho)}{d\rho}$ term.

The solution is a confluent hypergeometric function

$$F(\rho)={}_1F_1(l+1-d,2l+2,\rho) \tag{17}$$

This function blows up at large $\rho$ at a rate that cannot be compensated by the $e^{-\rho/2}$ in 15 unless its first argument is a negative integer or zero (this condition results from the series form of ${}_1F_1$ which must terminate). We therefore get the quantization condition

$$l+1-d=-n_r \tag{18}$$

where $n_r$ is a positive integer. Defining the principal quantum number $n$ as

$$n\equiv l+1+n_r \tag{19}$$

we can find the energy from 11 and 12.

$$d=\frac{Z\alpha}{2\sqrt{1-\frac{\varepsilon^2}{m_0^2c^4}}}=n \tag{20}$$

which gives the energies

$$\varepsilon=\pm\sqrt{1-\frac{Z^2\alpha^2}{4n^2}}m_0c^2 \tag{21}$$

The energy doesn't depend on the orbital angular momentum number $l$, but from 19 we see that since $n_r$ must be a positive integer, once we specify the principal quantum number $n$, then $l+n_r=n-1$, so $l=0,1,\ldots,n-1$ just as in the hydrogen atom from the Schrödinger equation.