CURL & POTENTIAL - EXAMPLE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 2.20.

An important consequence of the curl of the electric field being zero in electrostatics (that is, in the absence of any moving charges) comes from Stokes's theorem, which for the electric field is

$$\int_S (\nabla\times\mathbf{E})\cdot d\mathbf{a} = \int_L \mathbf{E}\cdot d\mathbf{l}$$

where the integral on the left is taken over some open surface S and that on the right is over the boundary curve L of that surface. Since the curl is zero, this means that the line integral of the electric field around any closed path is zero. This means that the line integral of $\mathbf{E}\cdot d\mathbf{l}$ along a path connecting two points a and b is independent of the path. This follows since if the integral did depend on the path, we could choose one path from a to b and then a different path (where the integral had a different absolute value) on the return path b to a. In that case the integral over the combined path, which is a closed curve, would not be zero, which isn't allowed.

In fact the path integral is another way of defining the electric potential function V: the potential difference between two points is the negative of the line integral of $\mathbf{E}\cdot d\mathbf{l}$ along any path connecting those two points:

$$V(\mathbf{b})-V(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}\cdot d\mathbf{l} \tag{1}$$

Although it is only the potential difference which has any physical significance, it is traditional to define the potential function so that it has a definite value at each point. To do this, we need to specify a reference point $\mathbf{p}$ at which $V(\mathbf{p})=0$ by definition, and then calculate all other values of the potential by integrating from that reference point to the point in question. For many applications, the reference point is chosen as infinity, because for any localized distribution of charge, its effect will fall off to zero as we get infinitely far from it. However, this isn't cast in stone, and in some problems, other reference points are more convenient.

The curl condition $\nabla\times\mathbf{E}=0$ means that we can't specify any old vector field as a description of an electric field in space. For example, if we tried to specify $\mathbf{E}$ as

$$\mathbf{E}=c\left[xy\hat{\mathbf{i}}+2yz\hat{\mathbf{j}}+3xz\hat{\mathbf{k}}\right]$$

we can prove this isn't a valid expression for an electric field by calculating its curl. We have

$$\nabla\times\mathbf{E}\equiv\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ cxy & 2cyz & 3cxz\end{vmatrix}$$

We need to work out only the x component to discover that it is $-2cy\hat{\mathbf{i}}\neq 0$, which means the curl cannot be zero so this is not a valid electric field.

However, the field given by

$$\mathbf{E}=c\left[y^2\hat{\mathbf{i}}+(2xy+z^2)\hat{\mathbf{j}}+2yz\hat{\mathbf{k}}\right]$$

is a valid electric field since $\nabla\times\mathbf{E}=0$ as can be verified by direct calculation.

We can work out the potential function for this field, but we need to specify a reference point. Although this is a valid electric field, it is a bizarre one since it gets larger the further from the origin we are. Since the field is zero at the origin, this seems a sensible place to set our reference point for the potential. We can use 1 to work out the potential, but we need a path of integration. The integral is independent of the path so it doesn't matter which path we choose. Since the field is specified in rectangular coordinates, a sensible path from the origin out to a point (x,y,z) seems to be as follows. Start at the origin and move along the x axis to the point $(x,0,0)$. Then from this point, keep x and z constant and follow the line from $(x,0,0)$ to $(x,y,0)$. Finally, keep x and y constant and move along the line from $(x,y,0)$ to (x,y,z).

Along the first line segment, $\mathbf{E}\cdot d\mathbf{l}=cy^2dx$, but since $y=0$ along our chosen path, the integral $\int_{(0,0,0)}^{(x,0,0)}cy^2dx'$ contributes zero.

Along the second line segment $\mathbf{E}\cdot d\mathbf{l}=c(2xy+z^2)dy=2xydy$ since $z=0$ along this segment. We get $\int_{(x,0,0)}^{(x,y,0)}2cxy'dy'=cxy^2$.

Finally along the third line segment $\mathbf{E}\cdot d\mathbf{l}=2cyzdz$ and $\int_{(x,y,0)}^{(x,y,z)}2cyz'dz'=cyz^2$.

The total potential is the sum of these three contributions, so we get

$$\begin{aligned} V(x,y,z) &= -(0+cxy^2+cyz^2) \\ &= -cxy^2-cyz^2 \end{aligned}$$

The answer can be verified directly by calculating $\mathbf{E}=-\nabla V$.