# CURVILINEAR COORDINATES

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In physics there are three commonly used coordinate systems: rectangular (also known as Cartesian), spherical and cylindrical. All three of these systems are special cases of what are known as *curvilinear* systems. In each case, a point in space is represented by three coordinates.

In rectangular coordinates, there are three mutually perpendicular axes labelled $x$, $y$ and $z$. Each axis has its own unit vector, usually denoted by $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ respectively. These three unit vectors are constants, since the alignment of the axes is constant over all space.

In spherical coordinates, a point is represented by its distance from the origin $r$, the angle $\theta$ that a vector to the point makes with the $z$ axis, and another angle $\phi$ that a projection of the radius vector into the $xy$ plane makes with the $z$ axis. The unit vectors in spherical coordinates are $\hat{\mathbf{r}}$, which points along the vector from the origin to the point, $\hat{\theta}$, which lies in the plane defined by $\hat{\mathbf{r}}$ and $\hat{\mathbf{k}}$ and is perpendicular to $\hat{\mathbf{r}}$, and finally $\hat{\phi}$ which is perpendicular to both $\hat{\mathbf{r}}$ and $\hat{\theta}$, so it lies in the $xy$ plane. All three of these vectors vary with the position being identified.

In cylindrical coordinates, a point is represented by its distance $r$ from the axis of the cylinder, the angle $\theta$ between the $x$ axis and the $r$ line, and the coordinate $z$ which is identical to the $z$ in rectangular coordinates. The unit vectors in this case are $\hat{\mathbf{r}}$, $\hat{\theta}$ and $\hat{\mathbf{z}}$, where $\hat{\mathbf{r}}$ and $\hat{\theta}$ vary with position, but $\hat{\mathbf{z}} = \hat{\mathbf{k}}$ always.

Suppose we wish to describe an infinitesimal line element $d\mathbf{l}$ in space using rectangular coordinates. In general, such an element starts at $(x,y,z)$ and ends at $(x+dx, y+dy, z+dz)$. The vector describing this element is then

$$d\mathbf{l} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}} \tag{1}$$

Each vector on the right hand side of this equation is the distance along one edge of a little cube.

Now suppose we want to describe the same element in spherical coordinates. We need to figure out what the distances along each side of the infinitesimal cube whose sides are parallel to the three spherical unit vectors (so it won't be the same cube as in the rectangular case) are in the spherical case. If we start at the coordinates $(r,\theta,\phi)$ and end at $(r+dr, \theta+d\theta, \phi+d\phi)$, then the edge in the $r$ direction has length $dr$. However, the length of the edge in the $\theta$ direction depends on $r$, since if we vary the angle by $d\theta$, this will move the point further if $r$ is larger (remember that, with an angle defined in radians, the length of a circular arc that subtends an angle $\theta$ is $r\times\theta$). Thus the side has length $r\,d\theta$.

Similarly for the angle $\phi$, since this angle is measured between the projection of the radius vector onto the $xy$ plane, we need to know the radius of that projection. This is $r\sin\theta$, so the length of the side of the cube in the $\phi$ direction is $r\sin\theta\;d\phi$.

Putting all this together, we get

$$d\mathbf{l} = dr\hat{\mathbf{r}} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi} \tag{2}$$

In a similar way, we get for cylindrical coordinates

$$d\mathbf{l} = dr\hat{\mathbf{r}} + rd\theta\hat{\theta} + dz\hat{\mathbf{z}} \tag{3}$$

In general if we have curvilinear coordinates whose three unit vectors are mutually perpendicular, we can write

$$d\mathbf{l} = f\,du\hat{\mathbf{u}} + g\,dv\hat{\mathbf{v}} + h\,dw\hat{\mathbf{w}} \tag{4}$$

Here $f$, $g$ and $h$ are functions of the three coordinates $u$, $v$ and $w$. In rectangular coordinates $u=x, v=y, w=z$; $f=g=h=1$, in spherical, $u=r, v=\theta, w=\phi$; $f=1, g=r, h=r\sin\theta$ and in cylindrical, $u=r, v=\theta, w=z$; $f=1, g=r, h=1$.

Using the same reasoning, we can see that the volume $d\tau$ of an infinitesimal cube in a curvilinear coordinate system is

$$d\tau = fgh\,dudvdw \tag{5}$$

The functions $f$, $g$ and $h$ also allow us to write general expressions for the most commonly used vector calculus operations: divergence, gradient and curl, but we'll leave those to other posts.

## PINGBACKS

Pingback: Gradient