GAUSS'S LAW

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Sec 2.2.

Gauss's law in electrostatics is a relation between the charge contained by a closed surface and the electric field that crosses that surface. The easiest way to see how it works is to begin with a point charge at the origin and a spherical surface centred at the origin. By symmetry the electric field due to the point charge is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \tag{1}$$

The *flux* of this field through the sphere is defined as the surface integral of the component of the field that is normal to the surface. That is the flux Φ is defined as

$$\Phi = \int \mathbf{E}\cdot d\mathbf{a} \tag{2}$$

where the integral extends over the surface, and $d\mathbf{a}$ is a differential vector whose magnitude is a differential area element and whose direction is normal to the surface at each point.

In the case of a sphere, it is not surprisingly easiest to use spherical coordinates, and in that case

$$d\mathbf{a} = r^2\sin\theta d\theta d\phi\hat{\mathbf{r}} \tag{3}$$

That is, the area element points radially outwards at each point on the sphere.

Combining these results, we see that for a point charge

$$\Phi = \int \mathbf{E} \cdot d\mathbf{a} \tag{4}$$
$$= \int_0^{2\pi} \int_0^{\pi} \left(\frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} \right) \cdot \left(r^2 \sin\theta d\theta d\phi \hat{\mathbf{r}} \right) \tag{5}$$
$$= \frac{q}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^{\pi} \sin\theta d\theta d\phi \tag{6}$$
$$= \frac{q}{\epsilon_0} \tag{7}$$

That is, the flux due to a point charge depends only on the magnitude of the charge and not on the radius of the sphere that contains it. This makes intuitive sense, since if we imagine a point charge 'emitting' the electric field, then as long as we provide a surface that wraps up the charge completely, the flux through that surface will be the same regardless of the size of the surface. In fact, it shouldn't depend on the *shape* of the surface either, so long as that surface completely encloses the charge. A more general derivation using an arbitrary closed surface is possible (using the solid angle subtended by the elemental area), but it adds nothing to the general idea.

From here, we can generalize the idea to a collection of point charges using the principle of superposition, and get, for a collection of n point charges, all enclosed by the surface:

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} \sum_{i=1}^{n} q_i \tag{8}$$

For a continuous charge distribution, where the charge density is $\rho(\mathbf{r})$, we get

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} \int \rho(\mathbf{r}) d^3\mathbf{r} \tag{9}$$

where it is important to note that the integral on the left is over the enclosing surface, while that on the right is over the volume enclosed by that surface. This is the integral form of Gauss's law for electrostatics.

Using the divergence theorem, we can equate the charge density with the divergence of the electric field:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{10}$$

This is the differential form of Gauss's law. Both these forms are very powerful in solving various types of problems since they allow electric fields to be calculated, often without requiring complicated integrals.

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