

VECTOR AREA

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 1.61.

This is a brief mathematical interlude since we'll need the results here when we discuss magnetic dipoles.

The *vector area* of a surface is the integral of the differential area vector over the surface. That is

$$(0.1) \quad \mathbf{a} \equiv \int_S d\mathbf{a}$$

Remember that $d\mathbf{a}$ is normal to the surface at each point.

As an example, we can calculate \mathbf{a} for a hemisphere of radius R . In spherical coordinates $|d\mathbf{a}| = R^2 \sin\theta d\theta d\phi$. Since the vector area definition is a vector equation, it's easiest to split it up into 3 equations in rectangular coordinates. The normal to the hemisphere is in the $\hat{\mathbf{r}}$ direction at every point, and

$$(0.2) \quad \hat{\mathbf{r}} = \sin\theta \cos\phi \hat{\mathbf{x}} + \sin\theta \sin\phi \hat{\mathbf{y}} + \cos\theta \hat{\mathbf{z}}$$

For the x component, we have

$$(0.3) \quad a_x = R^2 \int_0^{\pi/2} \int_0^{2\pi} \cos\phi \sin^3\theta d\phi d\theta$$

$$(0.4) \quad = 0$$

since the integral over ϕ gives zero. Similarly, $a_y = 0$. For a_z we get

$$(0.5) \quad a_z = R^2 \int_0^{\pi/2} \int_0^{2\pi} \cos\theta \sin\theta d\phi d\theta$$

$$(0.6) \quad = \pi R^2$$

Thus the vector area of a hemisphere has the same magnitude as the circle that is its base.

For a full sphere, we just extend the upper limit on θ to π in the above integrals, and we find that all three components are zero. In fact, this is a

special case of a more general theorem, which is that $\mathbf{a} = 0$ for *any* closed surface. To see this, we can apply the divergence theorem in the following way.

Suppose we have a vector field $\mathbf{v} = \mathbf{c}T$, where \mathbf{c} is a constant vector and T is some scalar field. Then if S is a closed surface and V is the volume it encloses:

$$(0.7) \quad \int_S \mathbf{v} \cdot d\mathbf{a} = \int_V \nabla \cdot \mathbf{v} d^3\mathbf{r}$$

$$(0.8) \quad \mathbf{c} \cdot \int_S T d\mathbf{a} = \int_V \nabla \cdot (\mathbf{c}T) d^3\mathbf{r}$$

$$(0.9) \quad = \int_V [\mathbf{c} \cdot \nabla T + T \nabla \cdot \mathbf{c}] d^3\mathbf{r}$$

$$(0.10) \quad = \mathbf{c} \cdot \int_V \nabla T d^3\mathbf{r}$$

where the last equality follows because \mathbf{c} is a constant. Since \mathbf{c} is arbitrary, the two integrals must be equal:

$$(0.11) \quad \int_S T d\mathbf{a} = \int_V \nabla T d^3\mathbf{r}$$

If we take $T = 1$ then the LHS is just \mathbf{a} and the RHS is zero since T is a constant. Thus the vector area of any closed surface is zero. This is because although the actual surface area is non-zero, the vector components always cancel each other out as we integrate over the surface.

Consider now a closed surface and divide it into two parts by cutting it along some closed curve L . Since the total vector area is zero, we must have $\mathbf{a}_{upper} = -\mathbf{a}_{lower}$. Now the shape of the surface is arbitrary, so for some curve L , we can keep the lower surface constant while varying the shape of the upper surface. That is, \mathbf{a}_{lower} is held constant while the upper surface varies. However, the equality of the two areas must always hold, so *any* surface enclosed by L must have the same vector area. This explains why the vector area of a hemisphere is just the area of the circle that defines its base: these are just two different surfaces sharing a boundary curve.

We can apply this to get a different formula for \mathbf{a} in terms of the boundary curve. Suppose we draw a cone with its vertex at the origin and with its base being the curve L . (The base need not lie in a plane, since L doesn't have to be flat. This won't affect the argument.) The vector area of this cone must be the same as any other surface that shares L . Now if we divide up L into line increments $d\mathbf{l}$ then we divide up the cone into a sequence of triangles

with sides \mathbf{r} (the vector from the vertex of the cone (the origin) to $d\mathbf{l}$), $d\mathbf{l}$ and $\mathbf{r} + d\mathbf{l}$. The area of this triangle is half the area of the parallelogram with two adjacent sides \mathbf{r} and $d\mathbf{l}$, and that in turn is the magnitude $\mathbf{r} \times d\mathbf{l}$. Therefore

$$(0.12) \quad \mathbf{a} = \frac{1}{2} \oint_L \mathbf{r} \times d\mathbf{l}$$

Finally, we can derive a similar result to the divergence one above using Stokes's theorem. Again using a vector field $\mathbf{v} = \mathbf{c}T$ we have

$$(0.13) \quad \int_L \mathbf{v} \cdot d\mathbf{l} = \int_S [\nabla \times (\mathbf{c}T)] \cdot d\mathbf{a}$$

$$(0.14) \quad \mathbf{c} \cdot \int_L T d\mathbf{l} = \int_S [T\nabla \times \mathbf{c} - \mathbf{c} \times \nabla T] \cdot d\mathbf{a}$$

$$(0.15) \quad = - \int_S (\mathbf{c} \times \nabla T) \cdot d\mathbf{a}$$

$$(0.16) \quad = -\mathbf{c} \cdot \int_S (\nabla T \times d\mathbf{a})$$

The last line uses the triple vector product identity $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$. Equating integrals gives

$$(0.17) \quad \int_L T d\mathbf{l} = - \int_S (\nabla T \times d\mathbf{a})$$

If we let $T = \mathbf{c} \cdot \mathbf{r}$ for a constant vector \mathbf{c} , we get

$$(0.18) \quad \int_L (\mathbf{c} \cdot \mathbf{r}) d\mathbf{l} = - \int_S \nabla (\mathbf{c} \cdot \mathbf{r}) \times d\mathbf{a}$$

$$(0.19) \quad = - \int_S [\mathbf{c} \times (\nabla \times \mathbf{r}) + (\mathbf{c} \cdot \nabla) \mathbf{r}] \times d\mathbf{a}$$

$$(0.20) \quad = - \int_S \mathbf{c} \times d\mathbf{a}$$

$$(0.21) \quad = -\mathbf{c} \times \mathbf{a} = \mathbf{a} \times \mathbf{c}$$

The second line omits the derivatives of \mathbf{c} which are all zero. To get the third line, we use $\nabla \times \mathbf{r} = 0$ and $(\mathbf{c} \cdot \nabla) \mathbf{r} = \mathbf{c}$ (both of which can be proved by direct calculation).

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