

## ELECTROSTATICS - LINEAR CHARGE DISTRIBUTIONS

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Sec 2.1.4, Problems 2.3 - 2.5.

When faced with a continuous distribution of charge, we can work out the electric field as a function of position by using integration instead of summation. In general, we have

$$(1) \quad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$

Here  $\mathbf{r}'$  is the position of volume element  $d^3\mathbf{r}'$  and  $\rho(\mathbf{r}')$  is the charge density at that point. There are three types of problems that occur commonly with continuous charge distributions: linear, surface and volume charges. We'll do a few examples using linear charges here to see how this works in practice. In many problems in electrostatics, it's advisable to make use of any symmetries that the configuration has.

### Example 1

Suppose we have a line segment extending along the  $x$  axis from  $-L$  to  $L$ . This line segment contains a constant linear charge density  $\lambda$  (measured in Coulombs/metre). What is the electric field at a point  $z$  on the  $z$  axis?

We can split the problem in two by solving for the  $x$  and  $z$  components of  $\mathbf{E}$  separately. Because the  $z$  axis divides the linear charge precisely in two, we can use symmetry to conclude that there is no net  $x$  component in the field. To work out the  $z$  component, we note that the contributions from  $+x$  and  $-x$  are equal.

In the formula above,  $\mathbf{r} - \mathbf{r}'$  is the vector from a point on the linear charge to the point  $z$  so we get

$$(2) \quad |\mathbf{r} - \mathbf{r}'| = \sqrt{x^2 + z^2}$$

The unit vector  $\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$  has a  $z$  component of  $z/\sqrt{x^2 + z^2}$  and the charge density is  $\rho = \lambda$  so we get for the magnitude of  $\mathbf{E}$  in the  $z$  direction:

$$(3) \quad E_z = \frac{1}{4\pi\epsilon_0} \int_0^L \frac{2\lambda z}{(x^2 + z^2)^{3/2}} dx$$

$$(4) \quad = \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z\sqrt{L^2 + z^2}}$$

where you can either work out the integral by hand or look it up or use software like Maple.

Note that for  $z \gg L$ , we get

$$(5) \quad E_z \rightarrow \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z^2}$$

which is equivalent to the field of a point charge  $q = 2\lambda L$  at a distance  $z$ .

For  $L \gg z$  we get

$$(6) \quad E_z \rightarrow \frac{1}{4\pi\epsilon_0} \frac{2\lambda}{z}$$

which is the formula for the field due to an infinitely long line of charge.

### Example 2

A slight variant on this problem is to remove one half of the line segment, so the linear charge now extends from  $x = 0$  to  $x = L$ , with the test point still on the  $z$  axis. The  $z$  component of the field will now be half that calculated above:

$$(7) \quad E_z = \frac{1}{4\pi\epsilon_0} \frac{\lambda L}{z\sqrt{L^2 + z^2}}$$

However, since the problem is no longer symmetric about the origin,  $E_x$  is no longer zero. The  $x$  component of  $\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$  is  $-x/\sqrt{x^2 + z^2}$  (the negative sign arises because the vector points *from*  $\mathbf{r}'$  *to*  $\mathbf{r}$ , and since all  $\mathbf{r}'$  locations are on the  $+x$  axis, and  $\mathbf{r}$  is on the  $z$  axis, the  $x$  component of  $\mathbf{r} - \mathbf{r}'$  is always negative) so we get

$$(8) \quad E_x = -\frac{1}{4\pi\epsilon_0} \int_0^L \frac{\lambda x}{(x^2 + z^2)^{3/2}} dx$$

$$(9) \quad = \frac{\lambda}{4\pi\epsilon_0} \frac{z - \sqrt{L^2 + z^2}}{z\sqrt{L^2 + z^2}}$$

For  $z \gg L$ :

$$(10) \quad E_z \rightarrow \frac{1}{4\pi\epsilon_0} \frac{\lambda L}{z^2}$$

$$(11) \quad E_x \rightarrow 0$$

### Example 3

We now have a square loop (like a wire bent into a square) lying in the  $xy$  plane with sides parallel to the axes and centred at the origin, and we want to find the field at some point  $z$  on the  $z$  axis. By symmetry, the field will be entirely along the  $z$  direction, and the contributions from all four sides will be equal. If we consider the edge where  $y = a/2$ , then the distance from a point on this edge to  $z$  is  $\sqrt{x^2 + \frac{a^2}{4} + z^2}$ . Using the same reasoning as in example 1, the  $z$  component of the unit vector connecting this point with  $z$  is  $z/\sqrt{x^2 + \frac{a^2}{4} + z^2}$  so we get

$$(12) \quad E_z = \frac{1}{4\pi\epsilon_0} \int_{-a/2}^{a/2} \frac{4\lambda z}{\left(x^2 + \frac{a^2}{4} + z^2\right)^{3/2}} dx$$

$$(13) \quad = \frac{1}{4\pi\epsilon_0} \frac{8\lambda az}{\sqrt{2a^2 + 4z^2} \left(z^2 + \frac{a^2}{4}\right)}$$

As a check, when  $z \gg a$ , we get

$$(14) \quad E_z \rightarrow \frac{1}{4\pi\epsilon_0} \frac{4\lambda a}{z^2}$$

which is the equivalent of the field due to a point charge  $q = 4\lambda a$ .

### Example 4

Now consider a circular loop of radius  $r$  in the  $xy$  plane, centred at the origin. By symmetry, the field is entirely in the  $z$  direction. A line segment on the circle has length  $rd\theta$ , where  $\theta$  is the angle in the  $xy$  plane. The contribution from all line segments is the same. The  $z$  component of the unit vector is  $z/\sqrt{z^2 + r^2}$  so the field is

$$(15) \quad E_z = \frac{1}{4\pi\epsilon_0} \frac{\lambda rz}{(z^2 + r^2)^{3/2}} \int_0^{2\pi} d\theta$$

$$(16) \quad = \frac{1}{4\pi\epsilon_0} \frac{2\pi r \lambda z}{(z^2 + r^2)^{3/2}}$$

## PINGBACKS

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- Pingback: [Electric potential from charges - Examples 1](#)
- Pingback: [Electric field & potential - more examples](#)
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