

## ELECTROSTATICS - SURFACE CHARGES

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problems 2.6 - 2.8.

Now a few examples of surface charge.

### Example 1

First, we consider a circular disk of radius  $R$  with surface charge density  $\sigma$  lying in the  $xy$  plane and centred at the origin. Find the electric field at a point on the  $z$  axis.

To solve this we can make use of the solution to the circular loop. In this case we're considering a circular ring of circumference  $2\pi r$  and thickness  $dr$ , so the amount of charge in the ring is  $2\pi r\sigma dr$  and from the earlier solution, the field due to this ring is

$$(0.1) \quad E_z = \frac{1}{4\pi\epsilon_0} \frac{2\pi r\sigma z}{(z^2 + r^2)^{3/2}} dr$$

To get the total field from the disk, we integrate over  $r$ :

$$(0.2) \quad E = \frac{1}{4\pi\epsilon_0} 2\pi\sigma z \int_0^R \frac{r}{(z^2 + r^2)^{3/2}} dr$$

$$(0.3) \quad = \frac{1}{4\pi\epsilon_0} 2\pi\sigma \frac{(\sqrt{z^2 + R^2} - z)}{\sqrt{z^2 + R^2}}$$

To get the limiting behaviours we can Taylor-expand the result. For  $z \gg R$ , we expand about  $R = 0$  and find the leading non-zero term is in  $R^2$ :

$$(0.4) \quad E \rightarrow \frac{1}{4\pi\epsilon_0} 2\pi\sigma \frac{1}{2z^2} R^2$$

$$(0.5) \quad = \frac{1}{4\pi\epsilon_0} \frac{\pi\sigma R^2}{z^2}$$

This is correct since the total charge on the disk is  $\sigma\pi R^2$  so the field is that due to a point charge of that amount.

If we let  $R \rightarrow \infty$  we get

$$(0.6) \quad E \rightarrow \frac{1}{4\pi\epsilon_0} 2\pi\sigma$$

$$(0.7) \quad = \frac{\sigma}{2\epsilon_0}$$

This is the field due to an infinite plane of charge. Note that the field is independent of  $z$  so is the same no matter how far away from the plane we are.

### Example 2

We have a spherical shell of charge with radius  $R$  and surface density  $\sigma$ , centred at the origin. Again, we seek the field at a point on the  $z$  axis.

Using spherical coordinates, a point on the sphere has coordinates  $(R, \theta, \phi)$  where  $\theta$  is the angle from the positive  $z$  axis, and  $\phi$  is the azimuthal angle. We can use the cosine law to write the distance between a point on the sphere and the field point:

$$(0.8) \quad |\mathbf{r} - \mathbf{r}'| = \sqrt{z^2 + R^2 - 2zR \cos \theta}$$

By symmetry, the field will again be in the  $z$  direction, so we need the  $z$  component of  $\mathbf{r} - \mathbf{r}'$ . To get this, we need the angle  $\alpha$  between  $\mathbf{r} - \mathbf{r}'$  and the  $z$  axis. To get this, project the point on the sphere onto the  $z$  axis; this gives a point with  $z$  coordinate  $R \cos \theta$ . The remaining distance along the  $z$  axis to the field point is therefore  $z - R \cos \theta$ , but this distance is the projection of  $\mathbf{r} - \mathbf{r}'$  onto the  $z$  axis. The cosine of the angle between  $\mathbf{r} - \mathbf{r}'$  and the  $z$  axis is this projection divided by  $|\mathbf{r} - \mathbf{r}'|$ , so we get

$$(0.9) \quad \cos \alpha = \frac{z - R \cos \theta}{|\mathbf{r} - \mathbf{r}'|}$$

$$(0.10) \quad = \frac{z - R \cos \theta}{\sqrt{z^2 + R^2 - 2zR \cos \theta}}$$

Now for a given value of  $\theta$ , we have a ring of charge with radius  $R \sin \theta$  and thickness  $R d\theta$  at  $z$  distance  $|\mathbf{r} - \mathbf{r}'|$  from the field point, so we can integrate over  $\theta$  to get the total field.

$$(0.11) \quad E = \frac{\sigma}{4\pi\epsilon_0} \int_0^\pi \frac{(z - R \cos \theta)(2\pi R \sin \theta)(R d\theta)}{(z^2 + R^2 - 2zR \cos \theta)^{3/2}}$$

This integral can be done using Maple, but there are two possibilities. First, if  $z > R$  so the field point is outside the sphere, we get

$$(0.12) \quad E = \frac{1}{4\pi\epsilon_0} \frac{4\pi R^2 \sigma}{z^2}$$

Since  $4\pi R^2 \sigma$  is the total charge on the sphere, we see that the sphere behaves like a point charge for all field points outside it.

Second, if  $z < R$  so we are inside the sphere, we get

$$(0.13) \quad E = 0$$

So anywhere inside a spherical shell with a uniform charge distribution, we feel no field at all.

### Example 3

The result of the last example can be used to find the field due to a sphere that contains a uniform volume charge density  $\rho$ . Since each spherical shell within the sphere behaves as a point charge to all points outside the shell, the field at a point outside the sphere ( $z > R$ ) is just

$$(0.14) \quad E = \frac{1}{4\pi\epsilon_0} \frac{4\pi R^3 \rho}{3z^2}$$

At a point inside the sphere, all shells outside the field point contribute nothing, so we get, for  $z < R$ :

$$(0.15) \quad E = \frac{1}{4\pi\epsilon_0} \frac{4\pi z^3 \rho}{3z^2}$$

$$(0.16) \quad = \frac{z\rho}{3\epsilon_0}$$

The field thus increases linearly within the sphere and then falls off as an inverse square outside.

### PINGBACKS

- Pingback: Electric potential from charges - Examples 1
- Pingback: Electrostatic boundary conditions - examples
- Pingback: Electrostatic pressure
- Pingback: Field of a polarized object - examples
- Pingback: Force between two sheets of current