

ELECTROSTATICS - SURFACE CHARGES

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problems 2.6 - 2.8.

Now a few examples of surface charge.

Example 1

First, we consider a circular disk of radius R with surface charge density σ lying in the xy plane and centred at the origin. Find the electric field at a point on the z axis.

To solve this we can make use of the solution to the circular loop. In this case we're considering a circular ring of circumference $2\pi r$ and thickness dr , so the amount of charge in the ring is $2\pi r\sigma dr$ and from the earlier solution, the field due to this ring is

$$(1) \quad E_z = \frac{1}{4\pi\epsilon_0} \frac{2\pi r\sigma z}{(z^2 + r^2)^{3/2}} dr$$

To get the total field from the disk, we integrate over r :

$$(2) \quad E = \frac{1}{4\pi\epsilon_0} 2\pi\sigma z \int_0^R \frac{r}{(z^2 + r^2)^{3/2}} dr$$

$$(3) \quad = \frac{1}{4\pi\epsilon_0} 2\pi\sigma \frac{(\sqrt{z^2 + R^2} - z)}{\sqrt{z^2 + R^2}}$$

To get the limiting behaviours we can Taylor-expand the result. For $z \gg R$, we expand about $R = 0$ and find the leading non-zero term is in R^2 :

$$(4) \quad E \rightarrow \frac{1}{4\pi\epsilon_0} 2\pi\sigma \frac{1}{2z^2} R^2$$

$$(5) \quad = \frac{1}{4\pi\epsilon_0} \frac{\pi\sigma R^2}{z^2}$$

This is correct since the total charge on the disk is $\sigma\pi R^2$ so the field is that due to a point charge of that amount.

If we let $R \rightarrow \infty$ we get

$$(6) \quad E \rightarrow \frac{1}{4\pi\epsilon_0} 2\pi\sigma$$

$$(7) \quad = \frac{\sigma}{2\epsilon_0}$$

This is the field due to an infinite plane of charge. Note that the field is independent of z so is the same no matter how far away from the plane we are.

Example 2

We have a spherical shell of charge with radius R and surface density σ , centred at the origin. Again, we seek the field at a point on the z axis.

Using spherical coordinates, a point on the sphere has coordinates (R, θ, ϕ) where θ is the angle from the positive z axis, and ϕ is the azimuthal angle. We can use the cosine law to write the distance between a point on the sphere and the field point:

$$(8) \quad |\mathbf{r} - \mathbf{r}'| = \sqrt{z^2 + R^2 - 2zR \cos \theta}$$

By symmetry, the field will again be in the z direction, so we need the z component of $\mathbf{r} - \mathbf{r}'$. To get this, we need the angle α between $\mathbf{r} - \mathbf{r}'$ and the z axis. To get this, project the point on the sphere onto the z axis; this gives a point with z coordinate $R \cos \theta$. The remaining distance along the z axis to the field point is therefore $z - R \cos \theta$, but this distance is the projection of $\mathbf{r} - \mathbf{r}'$ onto the z axis. The cosine of the angle between $\mathbf{r} - \mathbf{r}'$ and the z axis is this projection divided by $|\mathbf{r} - \mathbf{r}'|$, so we get

$$(9) \quad \cos \alpha = \frac{z - R \cos \theta}{|\mathbf{r} - \mathbf{r}'|}$$

$$(10) \quad = \frac{z - R \cos \theta}{\sqrt{z^2 + R^2 - 2zR \cos \theta}}$$

Now for a given value of θ , we have a ring of charge with radius $R \sin \theta$ and thickness $R d\theta$ at z distance $|\mathbf{r} - \mathbf{r}'|$ from the field point, so we can integrate over θ to get the total field.

$$(11) \quad E = \frac{\sigma}{4\pi\epsilon_0} \int_0^\pi \frac{(z - R \cos \theta)(2\pi R \sin \theta)(R d\theta)}{(z^2 + R^2 - 2zR \cos \theta)^{3/2}}$$

This integral can be done using Maple, but there are two possibilities. First, if $z > R$ so the field point is outside the sphere, we get

$$(12) \quad E = \frac{1}{4\pi\epsilon_0} \frac{4\pi R^2 \sigma}{z^2}$$

Since $4\pi R^2 \sigma$ is the total charge on the sphere, we see that the sphere behaves like a point charge for all field points outside it.

Second, if $z < R$ so we are inside the sphere, we get

$$(13) \quad E = 0$$

So anywhere inside a spherical shell with a uniform charge distribution, we feel no field at all.

Example 3

The result of the last example can be used to find the field due to a sphere that contains a uniform volume charge density ρ . Since each spherical shell within the sphere behaves as a point charge to all points outside the shell, the field at a point outside the sphere ($z > R$) is just

$$(14) \quad E = \frac{1}{4\pi\epsilon_0} \frac{4\pi R^3 \rho}{3z^2}$$

At a point inside the sphere, all shells outside the field point contribute nothing, so we get, for $z < R$:

$$(15) \quad E = \frac{1}{4\pi\epsilon_0} \frac{4\pi z^3 \rho}{3z^2}$$

$$(16) \quad = \frac{z\rho}{3\epsilon_0}$$

The field thus increases linearly within the sphere and then falls off as an inverse square outside.

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