GAUSS'S LAW - EXAMPLES

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problems 2.11 - 2.18.

Gauss's law in electrostatics relates the integral over a closed surface of the electric field to the integral over the enclosed volume of the charge density. That is

$$\int \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\epsilon_0}\int \rho(\mathbf{r})d^3\mathbf{r} \tag{1}$$

where it is important to note that the integral on the left is over the enclosing surface (often called the *Gaussian surface*), while that on the right is over the volume enclosed by that surface.

In certain rather specialized situations, Gauss's law allows the electric field to be found quite simply, without having to do sometimes horrendous integrals. The situations rely on the geometry of the charge distribution having some kind of symmetry. Here we'll give a few examples of how Gauss's law can be used in this way.

Example 1. We have a spherical shell with radius R and constant surface charge density σ. By taking a spherical Gaussian surface inside the shell, we see that $E = 0$ inside the shell, since there is no enclosed charge here. Outside the shell, we can take a spherical Gaussian surface with a radius $r > R$. Outside the shell $\mathbf{E}$ is radially symmetric. The magnitude can be found by integrating $\mathbf{E}\cdot d\mathbf{a}$ over the Gaussian surface:

$$\begin{aligned}
4\pi r^2 E &= \frac{q}{\epsilon_0} && (2)\\
&= \frac{4\pi R^2\sigma}{\epsilon_0} && (3)\\
E &= \frac{R^2\sigma}{\epsilon_0 r^2} && (4)
\end{aligned}$$

Thus outside the shell, the charge behaves as a point charge at the centre of the sphere.

Example 2. Now we take a sphere of radius R that has a uniform volume charge density ρ. For $r < R$

$$\begin{aligned}
\int \mathbf{E}\cdot d\mathbf{a} &= \frac{q}{\epsilon_0} && (5)\\
4\pi r^2 E &= \frac{4\pi r^3\rho}{3\epsilon_0} && (6)\\
E &= \frac{r\rho}{3\epsilon_0} && (7)
\end{aligned}$$

Outside the sphere, the sphere behaves as a point charge of magnitude $4\pi R^3\rho/3$ so

$$E = \frac{R^3\rho}{3\epsilon_0 r^2} \tag{8}$$

Example 3. For an infinitely long charged wire of linear charge density λ we can choose a cylindrical Gaussian surface of length L and radius s centred on the wire. By symmetry the field points radially away from the wire and the end caps contribute nothing. The enclosed charge is then $q = L\lambda$ and the integral over the cylindrical surface gives

$$\begin{aligned}
\int \mathbf{E}\cdot d\mathbf{a} &= \frac{L\lambda}{\epsilon_0} && (9)\\
2\pi sLE &= \frac{L\lambda}{\epsilon_0} && (10)\\
E &= \frac{\lambda}{2\pi s\epsilon_0} && (11)
\end{aligned}$$

Example 4. A sphere of radius R carries a volume charge density $\rho = kr$ where r is the distance from the centre and k is a constant. Inside the sphere, the enclosed charge as a function of r is

$$\begin{aligned}
q(r) &= 4\pi\int_0^r (kr')(r'^2)dr' && (12)\\
&= \pi kr^4 && (13)
\end{aligned}$$

Therefore, using Gauss's law, we get

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{\pi k r^4}{\epsilon_0} \tag{14}$$

$$4\pi r^2 E = \frac{\pi k r^4}{\epsilon_0} \tag{15}$$

$$E = \frac{k r^2}{4\epsilon_0} \tag{16}$$

Outside the sphere, the enclosed charge is $\pi k R^4$ so

$$E = \frac{\pi k R^4}{4\pi\epsilon_0 r^2} \tag{17}$$

$$= \frac{k R^4}{4\epsilon_0 r^2} \tag{18}$$

Example 5. A hollow spherical shell contains charge density $\rho = k/r^2$ for $a \leq r \leq b$. In the region $r < a$, $E = 0$ since again there is no enclosed charge. In the region $a \leq r \leq b$ we first calculate the enclosed charge.

$$q(r) = 4\pi \int_a^r \frac{k}{r'^2}(r'^2)dr' \tag{19}$$

$$= 4\pi k(r-a) \tag{20}$$

Gauss's law then says

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{q}{\epsilon_0} \tag{21}$$

$$4\pi r^2 E = \frac{4\pi k(r-a)}{\epsilon_0} \tag{22}$$

$$E = \frac{k(r-a)}{r^2\epsilon_0} \tag{23}$$

For $r > b$ we get

$$E = \frac{k(b-a)}{r^2\epsilon_0} \tag{24}$$

Example 6. A coaxial cable has a cylindrical inner core of radius a with uniform volume charge density ρ, and an outer cylindrical shell of radius b with a surface charge density that is of opposite sign to the charge on the core. The surface charge density is such that the cable is electrically neutral. Inside the inner cylinder, we can use the result of example 3. The field will point radially outward from the cylinder's axis. Since the volume charge density is ρ, the linear charge density for that portion of the cylinder inside radius s is $\pi s^2\rho$, so the field is

$$E = \frac{\pi s^2\rho}{2\pi s\epsilon_0} \tag{25}$$
$$= \frac{s\rho}{2\epsilon_0} \tag{26}$$

Between the inner cylinder and the outer shell, the linear charge density is $\pi a^2\rho$ so the field becomes

$$E = \frac{\pi a^2\rho}{2\pi s\epsilon_0} \tag{27}$$
$$= \frac{a^2\rho}{2s\epsilon_0} \tag{28}$$

Outside the outer shell, the total enclosed charge is zero since the cable is neutral, so $E=0$.

Example 7. An infinite plane slab has thickness $2d$, and carries a uniform volume charge density ρ. If the y axis is perpendicular to the plane and the plane $y=0$ is the centre plane of the slab, we can choose a Gaussian surface that is a cylinder of radius a with axis perpendicular to the slab and thickness $2y$. The enclosed charge is $\pi a^2(2y)\rho$, and by symmetry $\mathbf{E}$ points away from the slab on both sides and only contributes on the ends of the cylinder, so

$$\int \mathbf{E}\cdot d\mathbf{a} = \frac{q}{\epsilon_0} \tag{29}$$
$$E(2\pi a^2) = \frac{2\pi a^2\rho y}{\epsilon_0} \tag{30}$$
$$E = \frac{\rho y}{\epsilon_0} \tag{31}$$

Outside the slab

$$E = \frac{\rho d}{\epsilon_0} \tag{32}$$

That is, the electric field is constant no matter how far from the slab we are.

Example 8. We have two spheres, each of radius R, one of which has volume charge density $+\rho$ and the other of which has density $-\rho$. The vector from the centre of the positive sphere to the centre of the negative sphere is $\mathbf{d}$. The two spheres have a region of overlap and we want the electric field within this region.

We might be tempted to say that since, in the region of overlap, any volume contains zero net charge (since the densities are equal and opposite), there is zero field within this region. However, the problem with this argument is that when working out the surface integral of the field, there is no obvious symmetry we can invoke. Thus although it is correct to say that any integral $\int \mathbf{E}\cdot d\mathbf{a}$ over a closed surface entirely within the region of overlap is zero, this doesn't automatically translate to the field being zero as it did in earlier examples.

The problem does, however, have a simple solution. If we look back at example 2, we see that the electric field inside a uniformly positively charged sphere is (restoring the vector notation)

$$\mathbf{E}_r = \frac{\rho}{3\epsilon_0}\mathbf{r} \tag{33}$$

where $\mathbf{r}$ is the vector from the centre of the sphere to the point in question. Now suppose that $\mathbf{s}$ is the vector from the centre of the negative sphere to the same point. Because the charge is negative, we get

$$\mathbf{E}_s = -\frac{\rho}{3\epsilon_0}\mathbf{s} \tag{34}$$

so the total field is, using superposition

$$\begin{aligned}
\mathbf{E} &= \mathbf{E}_r + \mathbf{E}_s && (35)\\
&= \frac{\rho}{3\epsilon_0}(\mathbf{r}-\mathbf{s}) && (36)\\
&= \frac{\rho}{3\epsilon_0}\mathbf{d} && (37)
\end{aligned}$$

where $\mathbf{d}$ is the vector joining the two centres. Thus the field is constant in the region of overlap, although it is not zero.

This is a bit of a trick question, since it relies on the field being directly proportional to the radius vector. For other geometries, no such simple solution exists.

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