ELECTRIC POTENTIAL FROM CHARGES - EXAMPLES 1

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problems 2.25-2.26.

We've seen that the electric potential function can be defined in terms of the charge distribution:

$$V(\mathbf{r}) \equiv \frac{1}{4\pi\epsilon_0}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')d^3\mathbf{r}' \tag{1}$$

This is the more usual problem in calculating the potential, as opposed to the case we considered earlier where the potential is calculated from the electric field. Usually, the problem we wish to solve is: given a charge distribution, find the electric field. Although this can be solved by doing a vector integral directly, often finding the potential first and then using $\mathbf{E}=-\nabla V$ to find the field is easier.

Here we'll consider a few examples of calculating the field from the charge distribution.

Example 1. We have two point charges, each of charge $+q$ a distance d apart, so that they lie on the x axis at locations $x=\pm d/2$. Find the potential at a point on the z axis. In this case, we can use the discrete version of the potential formula, which can be obtained from 1 by using delta functions for the point charges, or else we can use a simple sum formula.

$$V = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{2}\frac{q_i}{|\mathbf{r}-\mathbf{r}_i|} \tag{2}$$

$$= \frac{2q}{4\pi\epsilon_0\sqrt{z^2+d^2/4}} \tag{3}$$

The field from this distribution along the z axis points in the z direction by symmetry, and is

$$\mathbf{E} = -\nabla V \tag{4}$$

$$= \frac{2qz}{4\pi\epsilon_0(z^2+d^2/4)^{3/2}}\hat{\mathbf{z}} \tag{5}$$

This answer agrees with that calculated directly. Note that we *cannot* calculate the field off the z axis from this potential formula, since at other locations the symmetry does not apply. We would need to calculate a more general formula for the potential first.

This is illustrated more clearly if we change one of the charges to $-q$. In that case, the above formula gives $V=0$ on the z axis. We can't infer the electric field from this however, since there is no longer any symmetry along the z axis. The field is, of course, not zero; rather it points in the x direction.

Example 2. We have a linear charge extending from $x=-L$ to $x=+L$ and wish to find the potential along the z axis. The potential due to an element of the linear charge of length dx is $(\lambda/4\pi\epsilon_0\sqrt{z^2+x^2})dx$ so the total potential is

$$V = \frac{2\lambda}{4\pi\epsilon_0}\int_0^L \frac{dx}{\sqrt{z^2+x^2}} \tag{6}$$

$$= \frac{2\lambda}{4\pi\epsilon_0}\left[\ln(L+\sqrt{z^2+L^2})-\ln z\right] \tag{7}$$

Since the configuration is again symmetric about the z axis, we can calculate the field

$$\mathbf{E} = -\nabla V \tag{8}$$

$$= \frac{2\lambda}{4\pi\epsilon_0}\left[-\frac{z}{\sqrt{z^2+L^2}\left(L+\sqrt{z^2+L^2}\right)}+\frac{1}{z}\right] \tag{9}$$

$$= \frac{2\lambda}{4\pi\epsilon_0}\left[\frac{L\sqrt{z^2+L^2}+L^2}{z\sqrt{z^2+L^2}\left(L+\sqrt{z^2+L^2}\right)}\right] \tag{10}$$

$$= \frac{2\lambda}{4\pi\epsilon_0}\frac{L}{z\sqrt{z^2+L^2}} \tag{11}$$

which agrees with Example 1 earlier.

Example 3. A flat circular disk of radius R lies in the xy plane with its centre at the origin. Find the potential on the z axis. The potential due to a ring of charge of thickness dr is $(2\pi\sigma r/4\pi\epsilon_0\sqrt{z^2+r^2})dr$ so the total potential from the disk is

$$V = \frac{\sigma}{2\epsilon_0}\int_0^R \frac{r}{\sqrt{z^2+r^2}}dr \tag{12}$$

$$= \frac{\sigma}{2\epsilon_0}\left[\sqrt{z^2+R^2}-|z|\right] \tag{13}$$

Due to the symmetry of the situation, the potential must be the same for $\pm z$, which is why we've used $|z|$ in the last line. This term comes from $\sqrt{z^2}$ so we need to decide which root to take in each case.

The distribution is again symmetric about the z axis, so the field is

$$\mathbf{E} = -\nabla V \tag{14}$$

$$= \frac{\sigma}{2\epsilon_0}\left[1-\frac{|z|}{\sqrt{z^2+R^2}}\right]\hat{\mathbf{z}} \tag{15}$$

which agrees with the Example 1 in an earlier post.

Example 4. Given a right circular cone with base radius R and height $h = R$, we can find the potential difference between the centre of the base and the tip of the cone. Let the cone's tip be at $z = 0$ and let the cone open upwards (like an ice-cream cone), so the centre of its base is at $z = R$. To calculate the potential, we can slice the cone horizontally into thin slabs, each of radius x at height z. Since the slant of the cone is constant, we have $z = x$ for each of these slabs. The area of the rim of each slab can be found as follows.

Since the cone's surface is slanted, it's not correct to say that the surface area of the rim of each slab is just $2\pi x dz$. Using Pythagoras, the actual surface area is $2\pi x\sqrt{(dx)^2+(dz)^2} = 2\pi x\sqrt{1+(dz/dx)^2}dx = 2\pi\sqrt{2}x dx$ since in this specialized case, $z = x$ for all points on the cone.

If the surface charge density is σ, then the amount of charge on the surface of a slab is $2\pi\sqrt{2}x\sigma dx$. For a general point z_V on the z axis, the potential of this ring of charge at height z is

$$dV = \frac{\sqrt{2}\sigma}{2\epsilon_0}\frac{x dx}{\sqrt{x^2+(z_V-z)^2}} \tag{16}$$

$$= \frac{\sqrt{2}\sigma}{2\epsilon_0}\frac{x dx}{\sqrt{x^2+(z_V-x)^2}} \tag{17}$$

where the last line uses $z = x$ for points on the cone's surface. Thus the total potential for the cone is

$$V=\frac{\sqrt{2}\sigma}{2\epsilon_0}\int_0^R\frac{xdx}{\sqrt{x^2+(z_V-x)^2}} \tag{18}$$

This integral can be evaluated using software, but it's not pretty. However, for the special cases of the centre of the base of the cone and the tip of the cone, the answer is a bit less messy. For the tip of the cone $z_V=0$ and we get

$$\begin{aligned} V(0) &= \frac{\sqrt{2}\sigma}{2\epsilon_0}\int_0^R\frac{dx}{\sqrt{2}} && (19)\\ &= \frac{\sigma R}{2\epsilon_0} && (20)\end{aligned}$$

For the centre of the base, $z_V=R$ and we get

$$\begin{aligned} V(R) &= \frac{\sqrt{2}\sigma}{2\epsilon_0}\int_0^R\frac{xdx}{\sqrt{x^2+(R-x)^2}} && (21)\\ &= \frac{\sqrt{2}\sigma}{2\epsilon_0}\frac{\sqrt{2}R}{4}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) && (22)\\ &= \frac{\sigma R}{4\epsilon_0}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) && (23)\end{aligned}$$

The potential difference is then

$$\begin{aligned} V(R)-V(0) &= \frac{\sigma R}{2\epsilon_0}\left[\frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)-1\right] && (24)\\ &= \frac{\sigma R}{2\epsilon_0}\left[\frac{1}{2}\ln\left((\sqrt{2}+1)^2\right)-1\right] && (25)\\ &= \frac{\sigma R}{2\epsilon_0}\left(\ln\left(\sqrt{2}+1\right)-1\right) && (26)\end{aligned}$$

In the second line, we multiplied the argument of the logarithm top and bottom by $\sqrt{2}+1$.

In order to calculate the field, we would need to evaluate the potential for general values of z_V and then calculate the gradient.

PINGBACKS

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