ELECTROSTATIC BOUNDARY CONDITIONS - EXAMPLES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 2.30.

We've seen that the electric field has a discontinuity of σ/ϵ_0 when we cross a surface charge of density σ, but that the potential is continuous. Here we offer some examples that verify this condition.

Example 1. From Example 1 in a previous post, we know that the electric field for an infinite plane is

$$\mathbf{E}=\frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}} \tag{1}$$

where σ is the surface charge density and $\hat{\mathbf{n}}$ is a unit vector normal to the plane, pointing away from the plane on both sides. The difference in field as we cross the plane is therefore σ/ϵ_0.

Example 2. If we have two infinite, parallel planes of charge, with one having a charge density of $+\sigma$ and the other of $-\sigma$, then by the principle of superposition, the field will be zero everywhere except between the planes, where it will be σ/ϵ_0 pointing from the positive plane to the negative plane. Crossing either plane thus gives rise to a discontinuity in the field of σ/ϵ_0.

Example 3. For a spherical shell of charge, from Example 1 in a previous post, we know that the field is $E=0$ inside the shell and points radially outward with a magnitude of

$$E=\frac{R^2\sigma}{\epsilon_0 r^2} \tag{2}$$

where R is the radius of the sphere and r is the distance from the centre. At the boundary, $r=R$ and $E=\sigma/\epsilon_0$ so the discontinuity across the shell is again σ/ϵ_0.

Example 4. Consider an infinitely long, hollow, cylindral tube with surface charge density σ. By symmetry, the field points radially outwards, so we can use Gauss's law to find it. If the radius of the tube is R and we consider a length L then if we consider a Gaussian cylinder enclosing this section, the total enclosed charge is $2\pi RL\sigma$. If the Gaussian cylinder has radius r its area (minus the end caps, which don't contribute) is $2\pi rL$ so

$$\oint \mathbf{E}\cdot d\mathbf{a} = \frac{q}{\epsilon_0} \tag{3}$$

$$2\pi rLE = \frac{2\pi RL\sigma}{\epsilon_0} \tag{4}$$

$$E = \frac{R\sigma}{r\epsilon_0} \tag{5}$$

At the boundary $r = R$ so the field outside is σ/ϵ_0. Inside, since there is no enclosed charge, $E = 0$ so again the discontinuity at the boundary is σ/ϵ_0.

Example 5. We can work out the potential due to a spherical shell of charge from the formula

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')d\mathbf{r}' \tag{6}$$

If we consider a sphere of radius R, and determine the potential at a point z on the z axis, then

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{z^2+R^2-2zr\cos\theta} \tag{7}$$

where θ is the angle between the z axis and the vector $\mathbf{r}'$. The integral we need to do in this case is

$$V(z) = \frac{2\pi\sigma}{4\pi\epsilon_0}\int_0^\pi \frac{R^2\sin\theta d\theta}{\sqrt{z^2+R^2-2zr\cos\theta}} \tag{8}$$

The value of this integral depends on the relative sizes of z and R. If $z < R$ so we are inside the sphere, then

$$V(z<R) = \frac{R\sigma}{\epsilon_0} \tag{9}$$

If $z > R$ so we are outside the sphere, then

$$V(z>R) = \frac{R^2\sigma}{z\epsilon_0} \tag{10}$$

At the boundary, $z = R$ and V is $R\sigma/\epsilon_0$ on both sides, so it is continuous. The gradient is zero inside the shell, while outside it is

$$-\nabla V = \frac{R^2\sigma}{z^2\epsilon_0} \tag{11}$$

so at the boundary $z=R$ and the gradient is σ/ϵ_0. Thus we get the discontinuity of the field as before.

PINGBACKS

Pingback: Electric field & potential - more examples