WORK AND ENERGY - POINT CHARGES

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Sec 2.4, Problem 2.31.

Since charges exert forces on each other through their electric fields, it will require the expenditure of energy, or work, to assemble any configuration of charges. Here we'll have a look at how much energy is required to assemble, and thus how much energy is stored, in a collection of discrete charges.

The force on a charge q due to an electric field $\mathbf{E}$ is $q\mathbf{E}$. From elementary physics, we know that the work done when an object is moved against a force is the negative (since we're opposing the force) of (force) times (distance). In general, if the force varies as a function of position, we get

$$W=-\oint\mathbf{F}\cdot d\mathbf{l} \tag{1}$$

where the integral is taken over the path through which the object is moved. The minus sign is an indication that we are opposing the force $\mathbf{F}$; if we work instead with the force that we must exert to move the object, then the minus sign is omitted.

For the electric field, then, we get

$$\begin{aligned} W &= -\int_{\mathbf{a}}^{\mathbf{b}}\mathbf{F}\cdot d\mathbf{l} && (2)\\ &= -q\int_{\mathbf{a}}^{\mathbf{b}}\mathbf{E}\cdot d\mathbf{l} && (3)\\ &= q(V(\mathbf{b})-V(\mathbf{a})) && (4) \end{aligned}$$

To get the last line, we've used the fact that, in electrostatics, the line integral of the electric field is independent of the path; it depends only on the endpoints **a** and **b**. We've seen earlier that the line integral of the field is the negative of the potential difference between the two endpoints.

So, in other words, the potential difference between two points is the work per unit charge required to move a charge between those two points. If we've set the reference point for the potential at infinity (that is, $V=0$ at infinity), then the work required to bring in a charge from infinity to a point $\mathbf{r}$ is

$$W = qV(\mathbf{r}) \tag{5}$$

We can apply this formula to find out how much energy is required to assemble a collection of point charges. To place a single charge q_1 at a location $\mathbf{r}_1$ takes no work, since there are no fields to work against. Bringing in a second charge q_2 requires working against the field due to q_1. The potential due to q_1 is $V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} q_1/|\mathbf{r}-\mathbf{r}_1|$, so if we want to place q_2 at position $\mathbf{r}_2$ the work required is

$$W_2 = \frac{1}{4\pi\varepsilon_0} q_2 \frac{q_1}{|\mathbf{r}_2-\mathbf{r}_1|} \tag{6}$$

Before we go any further, it's worth noting that this formula gives rise to a bit of a problem. What if we want to assemble a point charge itself? That is, suppose we want to build up a point charge of a certain size by bringing together other point charges and, in effect, gluing them together. This seems to be a valid procedure, since after all, if a charge is truly a mathematical point, we should be able to pile as many of these point charges on top of each other as we like without increasing the volume (that is, zero) occupied by the sum of all the charges.

However, if we try that, the above formula says this will require an infinite amount of work (since $\mathbf{r}_2 = \mathbf{r}_1$). This is, in fact, a recognized problem in electrodynamics, and the problems don't go away even in the quantum mechanical theory. In fact, we can't even get out of the problem by saying that there is no such thing as a point charge, since a lot of physicists think that the electron might actually be a point charge (at least its diameter, if it's non-zero, is so small that nobody has actually measured it yet).

With that caution in mind, let's ignore the problem and carry on. If we assume that the existence of point charges is possible (without taxing our minds as to how they are built), we can continue to add more point charges to our distribution. Adding a third charge q_3 at location $\mathbf{r}_3$ requires work

$$\begin{aligned} W_3 &= q_3 V_{1,2}(\mathbf{r}_3) &\quad (7)\\ &= \frac{1}{4\pi\varepsilon_0} q_3 \left[\frac{q_1}{|\mathbf{r}_3-\mathbf{r}_1|} + \frac{q_2}{|\mathbf{r}_3-\mathbf{r}_2|}\right] &\quad (8)\end{aligned}$$

The total work required to assemble all three charges is then

$$W_{1,2,3} = W_1 + W_2 + W_3 \tag{9}$$

$$= 0 + \frac{1}{4\pi\varepsilon_0} q_2 \frac{q_1}{|\mathbf{r}_2 - \mathbf{r}_1|} + \frac{1}{4\pi\varepsilon_0} q_3 \left[\frac{q_1}{|\mathbf{r}_3 - \mathbf{r}_1|} + \frac{q_2}{|\mathbf{r}_3 - \mathbf{r}_2|} \right] \tag{10}$$

The general pattern should be fairly obvious by now. To assemble n charges, the total work is

$$W = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} q_i \sum_{j=i+1}^{n} \frac{q_j}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{11}$$

If we extend the second sum to cover 1 through n (excluding $j = i$), we get

$$W = \frac{1}{2} \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} q_i \sum_{j=1, j\neq i}^{n} \frac{q_j}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{12}$$

$$= \frac{1}{2} \sum_{i=1}^{n} q_i V(\mathbf{r}_i) \tag{13}$$

where $V(\mathbf{r}_i)$ is the potential due to all the charges in the collection except q_i.

As an example, suppose we have arranged two charges of $-q$ at the ends of a diagonal in a square, and a charge of $+q$ on one of the other two corners of the square. How much work is required to bring in another charge of $+q$ from infinity and place it at the remaining corner of the square?

To work this out, we need to find the potential at this corner due to the existing three charges. If the length of each side of the square is a, then we get

$$W_4 = qV_{1,2,3}(\mathbf{r}_4) \tag{14}$$

$$= \frac{1}{4\pi\varepsilon_0} q \left[\frac{-2q}{a} + \frac{q}{\sqrt{2}a} \right] \tag{15}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{q^2}{a} \left[\frac{1}{\sqrt{2}} - 2 \right] \tag{16}$$

Once all four charges have been assembled, the total energy stored in the collection is

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i) \tag{17}$$

$$= \frac{1}{2}\frac{2q^2}{4\pi\varepsilon_0 a}\left[\frac{1}{\sqrt{2}} - 2 + \frac{1}{\sqrt{2}} - 2\right] \tag{18}$$

$$= \frac{q^2}{2\pi\varepsilon_0 a}\left[\frac{1}{\sqrt{2}} - 2\right] \tag{19}$$

In the second line, the potential at the location of one of the $-q$ charges is $\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{a} + \frac{q}{a} - \frac{q}{\sqrt{2}a}\right]$. Multiplying this by $-q$ gives the first two terms in the square brackets. Similar logic for one of the $+q$ charges gives the last two terms in the brackets. The factor of 2 in the numerator arises from the fact that there are two each of $-q$ and $+q$ charges.

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