CAPACITANCE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Sec. 2.5.4 & Problem 2.39

The electric potential is defined as

$$V(\mathbf{r}) \equiv \frac{1}{4\pi\varepsilon_0}\int \frac{1}{|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')d\mathbf{r}' \tag{1}$$

One of the properties of a conductor is that its surface is an equipotential; that is, the potential is the same everywhere on the surface of (and inside) a conductor. Although the calculation of the potential from the above formula for some geometric shape of conductor could be very difficult, we can see that if we change the charge density ρ by a constant factor everywhere, the potential will change by the same factor. That is, the potential is effectively proportional to the total charge.

The absolute value of the potential depends on the reference location we use; as we've seen before, it's traditional to choose $V = 0$ at infinity. However, the potential *difference* between two points does not depend on this reference location (since it cancels out when taking the difference). Since the potential itself is proportional to the total charge, so too is the potential difference. Thus if we arrange two conductors in some configuration, then there is a definite potential difference which is the same from any point on one conductor to any point on the other conductor.

As such, it makes sense to define a quantity called the *capacitance* which is the ratio of the charge on the conductors to the potential difference between them:

$$C \equiv \frac{Q}{V} \tag{2}$$

where Q is the amount of positive charge on one conductor (and $-Q$ is the amount of negative charge on the other conductor), and V is the potential difference between the two conductors, taken as (positive) minus (negative), so V is always positive. Thus C is always a positive quantity, and in SI units its unit is the farad, or coulomb per volt.

The name 'capacitance' can be thought of as the capacity of a system of two conductors for holding charge. The larger the capacitance, the more charge is required to produce a given potential difference. To calculate the capacitance we need to find an expression for the potential in terms of the amount of charge stored on the conductors.

Example 1. The parallel plate capacitor. If we have two parallel flat plates, each of area A and a distance d apart, what is the capacitance of the system? If the area is large compared to the separation, then we can approximate the field between the plates by σ/ε_0, where σ is the surface charge density: $\sigma = Q/A$. Since the field is therefore constant, the potential is found from an integral.

$$V = -\int \mathbf{E}\cdot d\mathbf{l} \tag{3}$$

$$= \frac{Q}{A\varepsilon_0}d \tag{4}$$

The capacitance is then

$$C = \frac{Q}{V} \tag{5}$$

$$= \frac{A\varepsilon_0}{d} \tag{6}$$

Thus the capacitance increases if we increase the area of the plates, or if we decrease the distance between them.

Example 2. If we now have two concentric conducting spheres with radii a and b ($a < b$), and place a charge $+Q$ on the inner sphere, we note that the field between the spheres is due entirely to the inner sphere, and is

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} \tag{7}$$

where $a < r < b$. The potential difference between the spheres is then found from

$$V = -\int \mathbf{E}\cdot d\mathbf{l} \tag{8}$$

$$= -\frac{Q}{4\pi\varepsilon_0}\int_a^b \frac{dr}{r^2} \tag{9}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) \tag{10}$$

$$= \frac{Q}{4\pi\varepsilon_0}\frac{b-a}{ab} \tag{11}$$

The capacitance is

$$\begin{aligned} C &= \frac{Q}{V} && (12) \\ &= 4\pi\varepsilon_0 \frac{ab}{b-a} && (13) \end{aligned}$$

Note the rather curious fact that this calculation doesn't depend at all on how much charge is on the outer sphere, although it does depend on where the outer sphere is (that is, on b). In fact, if the outer sphere goes off to infinity, the capacitance tends to

$$C \to 4\pi\varepsilon_0 a$$

Example 3. We have two concentric cylindrical conducting shells or radii a and b. Find the capacitance per unit length. First, we can find the field due to the inner cylinder. By symmetry, the field points radially outwards, so we can use Gauss's law to find it. If the radius of the cylinder is a and we consider a length L then if we consider a Gaussian cylinder enclosing this section, the total enclosed charge is $2\pi aL\sigma$. If the Gaussian cylinder has radius r its area (minus the end caps, which don't contribute) is $2\pi rL$ so

$$\begin{aligned} \oint \mathbf{E} \cdot d\mathbf{a} &= \frac{q}{\varepsilon_0} && (14) \\ 2\pi rLE &= \frac{2\pi aL\sigma}{\varepsilon_0} && (15) \\ E &= \frac{a\sigma}{r\varepsilon_0} && (16) \end{aligned}$$

The potential difference from $R = a$ to $R = b$ is therefore

$$\begin{aligned} V &= \frac{a\sigma}{\varepsilon_0} \int_a^b \frac{1}{r} dr && (17) \\ &= \frac{a\sigma}{\varepsilon_0} \ln \frac{b}{a} && (18) \end{aligned}$$

In a unit length, the charge is $Q = 2\pi a\sigma$, so

$$V = \frac{Q}{2\pi\varepsilon_0} \ln \frac{b}{a} \qquad (19)$$

and

$$C = \frac{Q}{V} \quad (20)$$

$$= \frac{2\pi\varepsilon_0}{\ln(b/a)} \quad (21)$$

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