ELECTRIC FIELD & POTENTIAL - MORE EXAMPLES

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problems 2.41, 2.42, 2.43, 2.44.

Here are a few more examples of the calculation of electric field and potential.

Example 1. Given a square sheet of charge with side length a and surface charge density σ, find the electric field at height z above the centre of the sheet.

From Example 3 in this post, the field at distance z above the centre of a square loop of charge of side length s and linear charge density λ is

$$E=\frac{1}{4\pi\epsilon_0}\frac{32s\lambda z}{\sqrt{2s^2+4z^2}\left(s^2+4z^2\right)} \tag{1}$$

If we think of this square loop as an element of the square sheet, where the thickness of the loop is $ds/2$ (so that the overall square has side length $s+ds$) then $\lambda=\sigma ds/2$ and the field of the sheet is

$$E=\frac{16\sigma z}{4\pi\epsilon_0}\int_0^a\frac{s}{\sqrt{2s^2+4z^2}\left(s^2+4z^2\right)}ds \tag{2}$$

The mathematical software (Maple, in this case) needs a bit of help with this integral, so we can use the substitution $u=s^2$; $du=2sds$ to get

$$\begin{aligned}
E &= \frac{8\sigma z}{4\pi\epsilon_0}\int_0^{a^2}\frac{du}{\sqrt{2u+4z^2}\left(u+4z^2\right)} && (3)\\
&= \frac{1}{4\pi\epsilon_0}\left[8\sigma\arctan\left(\frac{\sqrt{2a^2+4z^2}}{2z}\right)-2\pi\sigma\right] && (4)\\
&= \frac{2\sigma}{\pi\epsilon_0}\arctan\sqrt{1+a^2/2z^2}-\frac{\sigma}{2\epsilon_0} && (5)
\end{aligned}$$

The field points vertically upwards, by symmetry.

Example 2. If the electric field is given by

$$\frac{1}{r}\left(A\hat{\mathbf{r}}+B\sin\theta\cos\phi\hat{\phi}\right) \tag{6}$$

(A and B are constants), find the charge density.

In this case, we can use the differential form of Gauss's law: $\nabla\cdot\mathbf{E} = \rho/\epsilon_0$. In spherical coordinates, the divergence for a vector field with its θ component equal to zero is

$$\frac{\rho}{\epsilon_0} = \nabla\cdot\mathbf{E} \tag{7}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi} \tag{8}$$

$$= \frac{A}{r^2} - \frac{B\sin\phi}{r^2} \tag{9}$$

$$\rho = \frac{\epsilon_0}{r^2}\left(A - B\sin\phi\right) \tag{10}$$

Example 3. In a uniformly charged solid sphere, find the force of repulsion between two hemispheres.

To make the problem definite, we'll consider the 'north' and 'south' hemispheres, so the problem becomes one of finding the vertical force between these two hemispheres.

From Example 2 in a previous post, the electric field inside a uniformly charged sphere with density ρ is

$$E = \frac{r\rho}{3\epsilon_0} \tag{11}$$

The vertical force on a volume element is therefore

$$dF_z = \left(\rho d^3\mathbf{r}\right)\left(\frac{r\rho}{3\epsilon_0}\right)\cos\theta \tag{12}$$

So the total repulsive force on a hemisphere is

$$F_z = \frac{2\pi\rho^2}{3\epsilon_0}\int_0^R\int_0^{\pi/2} r^3\cos\theta\sin\theta d\theta dr \tag{13}$$

$$= \frac{\pi\rho^2}{3\epsilon_0}\frac{R^4}{4} \tag{14}$$

Using

$$\rho = \frac{3Q}{4\pi R^3} \tag{15}$$

we get

$$F_z = \frac{1}{4\pi\epsilon_0}\frac{3Q^2}{16R^2} \tag{16}$$

Note that we can't just find the repulsive force between two hemispheres of a spherical shell and then integrate over shells of sizes from zero up to the radius of the sphere because this doesn't take into account the force between hemispheres of different sizes.

Problem 4. A northern hemispherical shell of radius R has a uniform surface charge density of σ. Find the potential difference between the north pole and the centre of the base of the hemisphere.

For an inverted hemispherical bowl, we can use the method of Example 5 in this previous post, except the limits on the integral are now 0 to $\pi/2$. We therefore get

$$4\pi\epsilon_0 V(z) = \frac{2\pi R\sigma}{z}\left[\sqrt{R^2+z^2} - \sqrt{(R-z)^2}\right] \tag{17}$$

At the north pole, $z = R$ so

$$4\pi\epsilon_0 V(R) = 2\sqrt{2}\pi\sigma R \tag{18}$$

At the centre, $z = 0$ so we need to take a limit. We can rewrite the formula as

$$4\pi\epsilon_0 V(z) = \frac{2\pi R^2\sigma}{z}\left[\sqrt{1+z^2/R^2} - 1 + z/R\right] \tag{19}$$

The lowest order term in the expansion of the square root is

$$\sqrt{1+z^2/R^2} = 1 + \frac{1}{2}\frac{z^2}{R^2} \tag{20}$$

so the limit as $z \to 0$ is

$$4\pi\epsilon_0 V(0) = \lim_{z\to 0}\frac{2\pi R^2\sigma}{z}\frac{z}{R} \tag{21}$$
$$= 2\pi\sigma R \tag{22}$$

and the potential difference is

$$V(R) - V(0) = \frac{2\pi\sigma}{4\pi\epsilon_0}R\left(\sqrt{2}-1\right) \tag{23}$$
$$= \frac{R\sigma}{2\epsilon_0}\left(\sqrt{2}-1\right) \tag{24}$$