

## DIRAC DELTA FUNCTION IN THREE DIMENSIONS

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Sec. 1.5.3, Problem 2.46.

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.6.4.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

We've seen that we can define a curious function called the Dirac delta function in one dimension. Here we examine how this can be extended to three dimensions, and how this extension is relevant to electrostatics.

The easiest way to define a three-dimensional delta function is just to take the product of three one-dimensional functions:

$$(1) \quad \delta_3(\mathbf{r}) \equiv \delta(x)\delta(y)\delta(z)$$

The integral of this function over any volume containing the origin is again 1, and the integral of any function of  $\mathbf{r}$  is a simple extension of the one-dimensional case:

$$(2) \quad \int f(\mathbf{r})\delta_3(\mathbf{r}-\mathbf{a})d^3\mathbf{r} = f(\mathbf{a})$$

In electrostatics, there is one situation where the delta function is needed to explain an apparent inconsistency involving the divergence theorem. If we have a point charge  $q$  at the origin, the electric field of that charge is

$$(3) \quad \mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$

According to the divergence theorem, the surface integral of the field is equal to the volume integral of the divergence of that field:

$$(4) \quad \oint \mathbf{E} \cdot d\mathbf{a} = \int_V \nabla \cdot \mathbf{E} d^3\mathbf{r}$$

where the integral on the left is over some closed surface, and that on the

right is over the volume enclosed by the surface. In electrostatics, the integral on the right evaluates to the total charge contained in the volume divided by  $\epsilon_0$

$$(5) \quad \int_V \nabla \cdot \mathbf{E} d^3 \mathbf{r} = \frac{q}{\epsilon_0}$$

Now for the catch. If we calculate  $\nabla \cdot \mathbf{E}$  (in spherical coordinates) for the point charge, we get, since only the radial component of the field is non-zero:

$$(6) \quad \nabla \cdot \mathbf{E} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E_r)$$

$$(7) \quad = \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \frac{\partial(1)}{\partial r}$$

At this stage, we might be tempted to say that the derivative is zero (since the derivative of any constant is zero), but the problem is that at  $r = 0$  we also have a zero in the denominator, so we have the indeterminate fraction of zero-over-zero. Thus although it is true that  $\nabla \cdot \mathbf{E} = 0$  everywhere *except* the origin, we know from the divergence theorem that  $\int_V \nabla \cdot \mathbf{E} d^3 \mathbf{r} = \frac{q}{\epsilon_0}$  so we must have

$$(8) \quad \int_V \nabla \cdot \left( \frac{1}{r^2} \hat{\mathbf{r}} \right) d^3 \mathbf{r} = 4\pi$$

and

$$(9) \quad \nabla \cdot \left( \frac{1}{r^2} \hat{\mathbf{r}} \right) = 0 \quad \text{if } r \neq 0$$

These two conditions can be satisfied if

$$(10) \quad \nabla \cdot \left( \frac{1}{r^2} \hat{\mathbf{r}} \right) = 4\pi \delta_3(\mathbf{r})$$

Furthermore, since  $\nabla \frac{1}{r} = -\frac{1}{r^2} \hat{\mathbf{r}}$ , we have

$$(11) \quad \nabla^2 \frac{1}{r} = -4\pi \delta_3(\mathbf{r})$$

**Example.** Suppose we have some distribution of charge that gives a potential function

$$(12) \quad V(\mathbf{r}) = A \frac{e^{-\lambda r}}{r}$$

We can find the field by taking the gradient

$$(13) \quad \mathbf{E} = -\nabla V$$

$$(14) \quad = A \frac{e^{-\lambda r}}{r^2} (1 + \lambda r) \hat{\mathbf{r}}$$

We can now find the charge distribution by taking the divergence, remembering what we've discussed above. Applying the divergence formula in spherical coordinates directly gives

$$(15) \quad \rho = -\epsilon_0 A \left( \frac{\lambda^2 e^{-\lambda r}}{r} \right)$$

but this formula is valid only for  $r \neq 0$ . To get the full charge distribution we need to incorporate the delta function. Using the product rule for the divergence ( $\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$ ):

$$(16) \quad \nabla \cdot \mathbf{E} = \nabla \cdot \left[ \frac{\hat{\mathbf{r}}}{r^2} A e^{-\lambda r} (1 + \lambda r) \right]$$

$$(17) \quad = A e^{-\lambda r} (1 + \lambda r) \nabla \cdot \left( \frac{1}{r^2} \hat{\mathbf{r}} \right) + \frac{\hat{\mathbf{r}}}{r^2} \cdot \nabla \left( A e^{-\lambda r} (1 + \lambda r) \right)$$

$$(18) \quad = A e^{-\lambda r} (1 + \lambda r) (4\pi \delta_3(\mathbf{r})) - A \left( \frac{\lambda^2 e^{-\lambda r}}{r} \right)$$

$$(19) \quad = 4\pi A \delta_3(\mathbf{r}) - A \left( \frac{\lambda^2 e^{-\lambda r}}{r} \right)$$

$$(20) \quad \rho = A\epsilon_0 \left[ 4\pi \delta_3(\mathbf{r}) - \frac{\lambda^2 e^{-\lambda r}}{r} \right]$$

In the fourth line, we used the fact that  $f(\mathbf{r})\delta_3(\mathbf{r}) = f(0)\delta_3(\mathbf{r})$ , since the delta function is zero everywhere except at  $\mathbf{r} = 0$ .

From this, we can find the total net charge by integrating  $\rho$ :

$$(21) \quad Q = \int_V \rho d^3 \mathbf{r}$$

$$(22) \quad = A\epsilon_0 \int_V 4\pi \delta_3(\mathbf{r}) d^3 \mathbf{r} - A\epsilon_0 \int_V \frac{\lambda^2 e^{-\lambda r}}{r} d^3 \mathbf{r}$$

$$(23) \quad = 4\pi A\epsilon_0 - 4\pi A\epsilon_0 \lambda^2 \int_0^\infty \frac{e^{-\lambda r}}{r} r^2 dr$$

$$(24) \quad = 4\pi A\epsilon_0 \left( 1 - \frac{\lambda^2}{\lambda^2} \right)$$

$$(25) \quad = 0$$

That is, the delta function contributes a point charge of  $+4\pi A\epsilon_0$  at the origin, and the second term contributes a continuous charge distribution smeared out over all space that sums up to  $-4\pi A\epsilon_0$ .