POTENTIAL OF TWO CHARGED WIRES

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 2.47.

Two infinite wires lie in the x-y plane, parallel to the x axis. One carries a charge density of $+\lambda$ and lies at location $y=+a$ while the other carries a charge density of $-\lambda$ and lies at location $y=-a$. Find the potential at a location (x,y,z) in rectangular coordinates.

The field due to an infinite wire can be found using Gauss's law in cylindrical coordinates. For the wire carrying charge density $-\lambda$ we have, using a Gaussian cylinder of unit length centred on the wire (see Example 3 in this post)

$$2\pi r_{-}E_{-}=-\frac{\lambda}{\epsilon_0} \tag{1}$$

where r_{-} is the cylindrical distance from the wire, and for the wire with charge density $+\lambda$

$$2\pi r_{+}E_{+}=\frac{\lambda}{\epsilon_0} \tag{2}$$

The potentials from the two wires add (according to the superposition principle), so we get

$$V_{-} = -\int \mathbf{E}\cdot d\mathbf{l} \tag{3}$$

$$= \frac{\lambda}{2\pi\epsilon_0}\int_a^{s_{-}}\frac{dr_{-}}{r_{-}} \tag{4}$$

$$= \frac{\lambda}{2\pi\epsilon_0}\ln\frac{s_{-}}{a} \tag{5}$$

where the limits on the integral arise from taking the origin as the zero point for potential. The distance s_{-} is the cylindrical distance from the wire at which we want the potential.

Similarly, the potential for the positive wire is

$$V_+ = -\frac{\lambda}{2\pi\epsilon_0}\ln\frac{s_+}{a} \tag{6}$$

so the total potential is

$$V = \frac{\lambda}{2\pi\epsilon_0}\left[\ln\frac{s_-}{a} - \ln\frac{s_+}{a}\right] \tag{7}$$

$$= \frac{\lambda}{2\pi\epsilon_0}\ln\frac{s_-}{s_+} \tag{8}$$

In terms of (x,y,z) we have

$$s_- = \sqrt{(y+a)^2+z^2} \tag{9}$$

$$s_+ = \sqrt{(y-a)^2+z^2} \tag{10}$$

so we get

$$V = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{\sqrt{(y+a)^2+z^2}}{\sqrt{(y-a)^2+z^2}} \tag{11}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\ln\frac{(y+a)^2+z^2}{(y-a)^2+z^2} \tag{12}$$

We can find the equipotential surfaces, that is, the surfaces where $V = K$ for some constant K. First, note that if $y = 0$ then $V = 0$, so the xz plane is the equipotential surface for $V = 0$. To find the other surfaces, we can consider $V = K \neq 0$ so we have

$$\frac{4\pi\epsilon_0 K}{\lambda} = \ln\frac{(y+a)^2+z^2}{(y-a)^2+z^2} \tag{13}$$

$$e^{4\pi\epsilon_0 K/\lambda} = \frac{(y+a)^2+z^2}{(y-a)^2+z^2} \tag{14}$$

We can now define

$$A \equiv e^{4\pi\epsilon_0 K/\lambda} \tag{15}$$

so we get

$$A\left((y-a)^2+z^2\right) = (y+a)^2+z^2 \tag{16}$$

$$A\left(y^2-2ay+a^2+z^2\right) = y^2+2ay+a^2+z^2 \tag{17}$$

$$(A-1)y^2-(A+1)2ay+(A-1)a^2+(A-1)z^2 = 0 \tag{18}$$

$$y^2-\frac{A+1}{A-1}2ay+a^2+z^2 = 0 \tag{19}$$

$$\left(y-\frac{A+1}{A-1}a\right)^2+z^2 = \left(\frac{A+1}{A-1}a\right)^2-a^2 \tag{20}$$

$$= a^2\left(\frac{A^2+2A+1-A^2+2A-1}{(A-1)^2}\right) \tag{21}$$

$$= \frac{4a^2A}{(A-1)^2} \tag{22}$$

The fourth line is obtained by dividing through by $(A-1)$, which requires $A \neq 1$, or $K \neq 0$. We saw above that $K=0$ is a special case, giving the xz plane as a surface.

This equation has the form of a circle in the yz plane, with centre at $y = \frac{A+1}{A-1}a$ and $z=0$, and with a radius of $R = \sqrt{\frac{4a^2A}{(A-1)^2}}$. Thus the equipotential surfaces are circular cylinders, with axes given by the lines running through the centres of the circles. Note that since A was defined as an exponential, it is always positive, so the argument of the square root in the equation for the radius is always positive.

We can revert back to expressions containing K to see the relation between the surfaces and the potentials.

The radius becomes

$$R = \frac{2ae^{2\pi\epsilon_0 K/\lambda}}{e^{4\pi\epsilon_0 K/\lambda}-1} \tag{23}$$

$$= \frac{2a}{e^{2\pi\epsilon_0 K/\lambda}-e^{-2\pi\epsilon_0 K/\lambda}} \tag{24}$$

$$= \frac{a}{\sinh\left(2\pi\epsilon_0 K/\lambda\right)} \tag{25}$$

The axis is (with $z=0$):

$$y = \frac{e^{4\pi\epsilon_0 K/\lambda}+1}{e^{4\pi\epsilon_0 K/\lambda}-1}a \tag{26}$$

$$= \frac{e^{2\pi\epsilon_0 K/\lambda}+e^{-2\pi\epsilon_0 K/\lambda}}{e^{2\pi\epsilon_0 K/\lambda}-e^{-2\pi\epsilon_0 K/\lambda}}a \tag{27}$$

$$= \frac{\cosh\left(2\pi\epsilon_0 K/\lambda\right)}{\sinh\left(2\pi\epsilon_0 K/\lambda\right)}a \tag{28}$$

$$= \frac{a}{\tanh\left(2\pi\epsilon_0 K/\lambda\right)} \tag{29}$$

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