

## VACUUM DIODE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 2.48.

A vacuum diode is an electronic component consisting of two parallel plates called the cathode, maintained at potential zero, and the anode, at potential  $V_0$ . If the cathode is heated, the excited atoms within the plate emit electrons which are accelerated across the gap between the plates. After a short while, a steady state is reached in which a constant current  $I$  is maintained across the gap. If we assume that the area  $A$  of the plates is much greater than the square of the distance  $d$  between them, we can neglect edge effects and assume that all quantities are functions only of  $x$ , the location between the plates (with  $x = 0$  at the cathode and  $x = d$  at the anode). The relevant functions are the potential  $V(x)$ , the charge density between the plates  $\rho(x)$ , the speed of the electrons  $v(x)$  and the current  $I$ , which is a constant independent of  $x$  once a steady state has been reached. In addition, we can assume that once this steady state has been reached, the electron density at the cathode is such that the electric field is zero there (although its gradient is not).

From Gauss's law, we get Poisson's equation between the plates

$$\begin{aligned} (1) \quad \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\ (2) \quad \mathbf{E} &= -\nabla V \\ (3) \quad \nabla^2 V &= -\frac{\rho}{\epsilon_0} \end{aligned}$$

Since these functions depend only on  $x$ , we get

$$(4) \quad \frac{d^2 V}{dx^2} = -\frac{\rho}{\epsilon_0}$$

Since the force on an electron is  $q\mathbf{E}$  (remember  $q$  is negative for an electron), we can find the speed of an electron from the work done on it, which is translated into kinetic energy.

$$(5) \quad \frac{1}{2}mv^2 = -q \int_0^x \frac{dV}{dx'} dx'$$

$$(6) \quad = -qV(x)$$

$$(7) \quad v(x) = \sqrt{\frac{-2qV(x)}{m}}$$

Next, we can find the current  $I$  in terms of  $\rho$  and  $v$ . The current is the rate of flow of charge. Consider a thin slice (thickness  $dx$ ) of the space between the plates. The volume of this slice is  $A dx$  so the amount of charge contained in this slice is  $A\rho dx$ . If the charge in this slice takes time  $dt$  to pass a given point, then the current is

$$(8) \quad I = A\rho \frac{dx}{dt}$$

$$(9) \quad = A\rho(x)v(x)$$

so the charge density can be written as

$$(10) \quad \rho(x) = \frac{I}{Av(x)}$$

$$(11) \quad = \frac{I}{A} \sqrt{-\frac{m}{2qV(x)}}$$

Note that since the charge density arises from electrons, it is negative, so the current is also negative (current is usually defined as the flow of *positive* charge, so when we have a flow of electrons, the current is actually in the opposite direction to the flow of electrons).

Returning to Poisson's equation, we can now write a differential equation for the potential.

$$(12) \quad \frac{d^2V}{dx^2} = -\frac{\rho}{\epsilon_0}$$

$$(13) \quad = -\frac{I}{A\epsilon_0} \sqrt{-\frac{m}{2q}} V^{-1/2}$$

$$(14) \quad V^{1/2}V'' = C$$

$$(15) \quad C \equiv -\frac{I}{A\epsilon_0} \sqrt{-\frac{m}{2q}}$$

Again, note that  $C$  is positive and real, since both  $I$  and  $q$  are negative. We can solve this by guessing a solution of form

$$(16) \quad V = Bx^k$$

where  $B$  and  $k$  are to be determined. Note that this form satisfies the boundary conditions  $V(0) = V'(0) = 0$ , which we require from above (the potential at the cathode is zero, and at steady state, the electric field ( $= -V'(0)$ ) is also zero). We just have to hope that it also provides a solution to the differential equation.

From this we get

$$(17) \quad V' = kBx^{k-1}$$

$$(18) \quad V'' = k(k-1)Bx^{k-2}$$

$$(19) \quad B^{1/2}x^{k/2}k(k-1)Bx^{k-2} = C$$

$$(20) \quad B^{3/2}k(k-1)x^{3k/2-2} = C$$

Since the RHS is a constant, the LHS must have the same value for all values of  $x$ , so the exponent of  $x$  must be zero. That is,

$$(21) \quad \frac{3k}{2} - 2 = 0$$

$$(22) \quad k = \frac{4}{3}$$

This allows us to find  $B$ :

$$(23) \quad \frac{4}{9}B^{3/2} = C$$

$$(24) \quad B = \left(\frac{9}{4}C\right)^{2/3}$$

So we get

$$(25) \quad V(x) = \left(\frac{9}{4}C\right)^{2/3} x^{4/3}$$

$$(26) \quad = \left[-\frac{9I}{4A\epsilon_0} \sqrt{-\frac{m}{2q}}\right]^{2/3} x^{4/3}$$

$$(27) \quad = \left(\frac{81I^2m}{-32A^2\epsilon_0^2q}\right)^{1/3} x^{4/3}$$

In particular, the potential at the anode is

$$(28) \quad V_0 = V(d) = \left( \frac{81I^2m}{-32A^2\epsilon_0^2q} \right)^{1/3} d^{4/3}$$

$$(29) \quad V(x) = V_0 \left( \frac{x}{d} \right)^{4/3}$$

Note that this isn't linear, as it would be if there were no charge between the plates.

From this, we can write the current in terms of the anode's potential

$$(30) \quad I = -\frac{4A\epsilon_0}{9d^2} \left( \frac{-2q}{m} \right)^{1/2} V_0^{3/2}$$

(we've taken the negative root, since  $I$  is negative). This relation between current and potential difference in a diode is known as the Child-Langmuir law.

The charge density

$$(31) \quad \rho(x) = \frac{I}{A} \sqrt{-\frac{m}{2qV(x)}}$$

$$(32) \quad = -\frac{4\epsilon_0V_0}{9d^{4/3}x^{2/3}}$$

Finally, the speed of the electrons is

$$(33) \quad v(x) = \sqrt{\frac{-2qV(x)}{m}}$$

$$(34) \quad = \sqrt{\frac{-2qV_0}{m}} \left( \frac{x}{d} \right)^{2/3}$$