DEVIATION FROM COULOMB'S LAW

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 2.49.

An interesting alternative (and fictional) world scenario for electrostatics arises if we alter the spatial dependence of Coulomb's law so that the force between two charges is

$$\mathbf{F}=\frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{|\mathbf{r}-\mathbf{r}'|^3}\left(1+\frac{|\mathbf{r}-\mathbf{r}'|}{\lambda}\right)e^{-|\mathbf{r}-\mathbf{r}'|/\lambda}\left(\mathbf{r}-\mathbf{r}'\right) \tag{1}$$

where λ is some (very large) length. We can see that if $\lambda\to\infty$ this formula reduces to the normal Coulomb force law. If we assume that the principle of superposition still holds, we can derive a few analogs to the electrostatic equations we all know and love.

First, the electric field due to a point charge q at position $\mathbf{r}'$ is

$$\mathbf{E}(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\frac{q}{|\mathbf{r}-\mathbf{r}'|^3}\left(1+\frac{|\mathbf{r}-\mathbf{r}'|}{\lambda}\right)e^{-|\mathbf{r}-\mathbf{r}'|/\lambda}\left(\mathbf{r}-\mathbf{r}'\right) \tag{2}$$

from which we can get the field due to a charge distribution

$$\mathbf{E}(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\int_V\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\left(1+\frac{|\mathbf{r}-\mathbf{r}'|}{\lambda}\right)e^{-|\mathbf{r}-\mathbf{r}'|/\lambda}\left(\mathbf{r}-\mathbf{r}'\right)d^3\mathbf{r}' \tag{3}$$

Although this looks ugly, we notice that a potential function exists for this field by using the following argument. First we consider a point charge at the origin, for which

$$\mathbf{E}(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda}\hat{\mathbf{r}} \tag{4}$$

Since the field depends only on r and is radial in direction, we can write, in spherical coordinates

$$\mathbf{E}\cdot d\mathbf{l}=\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda}dr \tag{5}$$

Therefore, the line integral of the field between two points **a** and **b** is

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = \frac{q}{4\pi\varepsilon_0} \int_{\mathbf{a}}^{\mathbf{b}} \left(\frac{e^{-r/\lambda}}{r^2} + \frac{e^{-r/\lambda}}{\lambda r} \right) dr \tag{6}$$

$$= \frac{q}{4\pi\varepsilon_0} \left(\frac{e^{-r_a/\lambda}}{r_a} - \frac{e^{-r_b/\lambda}}{r_b} \right) \tag{7}$$

where r_a is the distance from the origin to point **a** (similarly for r_b). Clearly if the path is closed, then $r_a = r_b$ and the integral is zero, so by Stokes's theorem, $\nabla \times \mathbf{E} = 0$, which is the same result as in normal electrostatics. Also, the integral above must be path independent, since otherwise we could choose a closed loop where the integrals along two branches of the loop were different, and thus the integral over the whole loop was non-zero, contradicting what we've just said. This means that, just as with normal electrostatics, we can define a potential relative to some reference point **x** as

$$V(\mathbf{r}) \equiv -\int_{\mathbf{x}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} \tag{8}$$

If we pick infinity as the reference point, then we get, for a point charge at the origin:

$$V(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0} \frac{e^{-r/\lambda}}{r} \tag{9}$$

Also for a point charge at the origin, we can integrate the field over a sphere of radius R centred at the origin:

$$\oint_R \mathbf{E} \cdot d\mathbf{a} = \frac{4\pi R^2}{4\pi\varepsilon_0} \frac{q}{R^2} \left(1 + \frac{R}{\lambda}\right) e^{-R/\lambda} \tag{10}$$

$$= \frac{q}{\varepsilon_0} \left(1 + \frac{R}{\lambda}\right) e^{-R/\lambda} \tag{11}$$

This isn't quite as nice as Gauss's law because of the extra R-dependent factors, but if we work out the volume integral of the potential over the same sphere:

$$\int V d^3\mathbf{r} = \frac{q}{4\pi\varepsilon_0}\int_0^R \frac{e^{-r/\lambda}}{r} r^2 dr \tag{12}$$

$$= \frac{q\lambda^2}{\varepsilon_0}\left(1 - e^{-R/\lambda}\left(1+\frac{R}{\lambda}\right)\right) \tag{13}$$

From these two results, we see that

$$\oint_R \mathbf{E}\cdot d\mathbf{a} + \frac{1}{\lambda^2}\int V d^3\mathbf{r} = \frac{q}{\varepsilon_0} \tag{14}$$

Remember that the first integral is a surface integral and the second integral is a volume integral. This is a 'sort-of' Gauss's law for the fictional electrostatics.

In fact, this argument applies to any shape of enclosing volume. Suppose we take an infinitesimal patch on the sphere, defined by angle increments $d\theta$ and $d\phi$. Then for that patch we have the increment in the surface integral:

$$\Delta\left(\oint \mathbf{E}\cdot d\mathbf{a}\right) = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}\left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}R^2\sin\theta d\theta d\phi \tag{15}$$

$$= \frac{q}{4\pi\varepsilon_0}\left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}\sin\theta d\theta d\phi \tag{16}$$

For the volume integral extending from the centre of the sphere up to the patch we need integrate over r only to get the increment in this integral

$$\Delta\left(\frac{1}{\lambda^2}\int V d^3\mathbf{r}\right) = \frac{1}{\lambda^2}\frac{q}{4\pi\varepsilon_0}\left[\int_0^R \frac{e^{-r/\lambda}}{r}r^2 dr\right]\sin\theta d\theta d\phi \tag{17}$$

$$= \frac{q}{4\pi\varepsilon_0}\left(1 - e^{-R/\lambda}\left(1+\frac{R}{\lambda}\right)\right)\sin\theta d\theta d\phi \tag{18}$$

The sum of these two terms comes out to

$$\Delta\left(\oint \mathbf{E}\cdot d\mathbf{a}\right) + \Delta\left(\frac{1}{\lambda^2}\int V d^3\mathbf{r}\right) = \frac{q}{4\pi\varepsilon_0}\sin\theta d\theta d\phi \tag{19}$$

That is, the dependence on the radius of the sphere cancels out for each incremental patch. This means we can consider any shape composed of incremental patches, where the radius of each patch can be different, which amounts to saying that we can consider an enclosing surface of any shape we like, and the same result holds. Further, by the principle of superposition, we can combine the results from any collection of charges so that we can write

$$\oint_R \mathbf{E}\cdot d\mathbf{a} + \frac{1}{\lambda^2}\int V d^3\mathbf{r} = \frac{Q}{\varepsilon_0} \tag{20}$$

where the first integral is over any enclosing surface and the second integral is over the enclosed volume, and Q is the total charge enclosed by the surface.

Although this result appears reasonably pretty, it does depend on the specific nature of the field equation (that is, the pair of extra factors that have been inserted). If we considered a different deviation from the classic Coulomb's law, these results wouldn't follow. But then, there's no evidence that there is *any* deviation from Coulomb's law anyway, so the whole exercise is just for fun.