LAPLACE'S EQUATION - AVERAGE VALUES OF SOLUTIONS

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Sec 3.1, Problems 2.50, 3.1.

We've seen that the electric field obeys Gauss's law, which in differential form is

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\varepsilon_{0}} \tag{1}$$

The field can also be written as the gradient of a potential function, so we get

$$\begin{aligned}\mathbf{E} &= -\nabla V &(2)\\ \nabla^{2}V &= -\frac{\rho}{\varepsilon_{0}} &(3)\end{aligned}$$

The last equation is a partial differential equation (PDE) known as Poisson's equation, and its solution gives the potential for a given charge distribution.

In regions where there is no charge, $\rho=0$ and Poisson's equation becomes Laplace's equation:

$$\nabla^{2}V=0 \tag{4}$$

One of the key points in the solution of any PDE is the specification of boundary conditions. Without boundary conditions, the problem is incompletely specified, and some surprising results can occur. For example, if we specify an electric field by the conditions (where a is a constant):

$$\begin{aligned}E_{x} &= ax &(5)\\ E_{y}=E_{z} &= 0 &(6)\end{aligned}$$

then we can verify by direct calculation that $\nabla\times\mathbf{E}=0$ as required in electrostatics, so it looks like this is an acceptable field. However if we calculate ρ we get

$$\nabla \cdot \mathbf{E} = a \tag{7}$$

$$\rho = a\varepsilon_0 \tag{8}$$

That is, the charge density is constant over all space. We get a paradoxical situation where the charge is uniform and yet the electric field has a specific direction. Clearly we'd get similar results if we specified a field by equations like $E_x = E_z = 0; \quad E_y = by$ or $E_x = E_y = 0; \quad E_z = cz$. Poisson's equation for the potential becomes, in this case

$$\nabla^2 V = -a \tag{9}$$

There are one obvious solution to this:

$$V = -\frac{a}{2}x^2 \tag{10}$$

which returns the original electric field when the gradient is taken, but again seems paradoxical since the potential for a uniform charge distribution has a dependence on x.

The problem is that although the div and curl equations are necessary for an electric field, they are not sufficient to determine the problem we're trying to solve. We need to specify the region of space in which the field has the given form, since the equations a given result in an infinite field as $x \to \infty$. We've solved field and potential problems earlier in which there is a uniform charge density, but it is restricted to a certain volume, such as a sphere.

But getting back to Laplace's equation, there are a couple of important properties possessed by all solutions to the equation. Proving these things rigorously takes a bit of heavy mathematics, but we can get a feel for these properties by some relatively simple calculations. The first property is that if we have a solution of Laplace's equation in three dimensions, then if we consider the value of the solution at a given point $\mathbf{r}$, then if we calculate the average value of the solution over any sphere centred at $\mathbf{r}$, the value of this average is equal to the value of the solution at $\mathbf{r}$. That is if we integrate V over the surface of a sphere of radius R centred at $\mathbf{r}$, we must have

$$\frac{1}{4\pi R^2}\oint_R V\, da = V(\mathbf{r}) \tag{11}$$

This is the main result which requires a bit of heavy-duty math to prove in general, but once we have established this fact, the second property of solutions to Laplace's equation follows quite easily: all extreme values (maxima and minima) of a solution must occur on the boundaries of the region under consideration. This is fairly obvious, since if an extremum occurred at some interior point $\mathbf{r}_0$ then we could find some sphere around this point where the values of V are all greater than (for a minimum) or less than (for a maximum) the value at $\mathbf{r}_0$, which isn't allowed according to the first property.

We can demonstrate this first property for the special case of the potential due to a point charge. For regions not containing the point charge, Laplace's equation is satisfied (since there is no charge there), and we've seen earlier that the potential due to a point charge q at location $\mathbf{r}'$ is

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\mathbf{r}-\mathbf{r}'|} \tag{12}$$

Thus we know that $V(\mathbf{r})$ is a solution to Laplace's equation everywhere away from the point charge. We can therefore try to show the averaging property for this particular solution.

To make things definite, suppose the charge is located at position z on the z axis, and we calculate the average value of V over a sphere of radius R centred at the origin. We'll make $z > R$ to ensure that the charge is outside the sphere, so Laplace's equation is valid everywhere within and on the surface of the sphere.

From the cosine law, a point on the surface of the sphere will satisfy

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{z^2+R^2-2Rz\cos\theta} \tag{13}$$

where θ is the angle between the z axis and the vector pointing to the point on the sphere. Using spherical coordinates, we can calculate the average of V:

$$\frac{1}{4\pi R^2}\oint_R V\,da = \frac{1}{4\pi R^2}\frac{q}{4\pi\varepsilon_0}\int_0^{\pi}\int_0^{2\pi}\frac{R^2\sin\theta}{\sqrt{z^2+R^2-2Rz\cos\theta}}d\phi\, d\theta \tag{14}$$

$$= \frac{1}{8\pi}\frac{q}{\varepsilon_0}\left[\frac{1}{zR}\sqrt{(z+R)^2}-\sqrt{(z-R)^2}\right] \tag{15}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{z} \tag{16}$$

It is important to note that in going from the second to the third line, we used $\sqrt{(z-R)^2} = +(z-R)$, since $z > R$.

The final result is just the potential due to the charge at the origin, that is, at the centre of the sphere, so in this case, the average of the potential over the sphere *is* equal to its value at the centre.

This result can be generalized by using the superposition principle to show that the average, over a sphere in a region where Laplace's equation is satisfied, of a potential due to any distribution of charge is equal to the potential at the centre of the sphere. Thus we've established the averaging property of Laplace's equation for electrostatic potential, but this isn't, of course, a general demonstration that *all* solutions of Laplace's equation satisfy the property.

We can note that the averaging formula can also be generalized to include charge that is inside the sphere. In the above derivation, if we consider a charge that is inside the sphere, then $z<R$ and we need to take the other root in the last step: $\sqrt{(z-R)^2}=-(z-R)$. This gives the average potential as

$$\frac{1}{4\pi R^2}\oint_R V\,da=\frac{q}{4\pi\varepsilon_0 R} \tag{17}$$

That is, the radius R has replaced z in the denominator. Using the superposition principle again, we get a general formula for the average potential

$$\bar{V}=\bar{V}_{out}+\frac{Q_{in}}{4\pi\varepsilon_0 R} \tag{18}$$

where $\bar{V}_{out}$ is the average of the potential due to all charge outside the sphere (or, what is the same thing, the potential at the centre of the sphere due to all charge outside the sphere), and Q_{in} is the total charge enclosed by the sphere.

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