METHOD OF IMAGES

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Sec 3.2, Problem 3.6.

We've seen that, in electrostatics, Laplace's equation $\nabla^2 V = 0$ governs the potential in those regions where there is no charge. Laplace's equation is a special case of Poisson's equation:

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \tag{1}$$

where V is the potential and ρ is the volume charge density. We've also seen that, given a particular geometry of bounding surfaces and boundary conditions specified on those surfaces, the solution of Laplace's equation is unique. This allows a clever trick to be used in some situations. The trick is known as the *method of images*. There are two standard problems that are usually given in textbooks to illustrate the method of images; we'll have a look at the first one here.

The defining problem for the method of images is the point charge $+q$ and the infinite, grounded conducting plane. To make things definite, we'll suppose the conducting plane occupies the xy plane, and the point charge is at location $z = d$ on the z axis. In this case, the boundary condition is that $V = 0$ on the xy plane (a *grounded* conductor is always assumed to be at zero potential). The problem here is that, since the plane is a conductor, charge is free to move around on its surface in response to the electric field from the point charge. It seems clear that this charge will have radial symmetry about the z axis, but beyond that, it's hard to tell precisely what the distribution will be.

The trick is to notice that if we replace the conducting plane by another point charge $-q$ at location $z = -d$. With just two point charges, we can write down the potential right away. In rectangular coordinates, we get

$$V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right] \tag{2}$$

The clever bit is to notice that in the xy plane (at $z = 0$), $V = 0$. Thus we have a potential that satisfies Laplace's equation (at least at every point except where the two point charges are), and also satisfies the boundary condition that $V = 0$ on the xy plane. Since we know that solutions to Laplace's equation are unique, this must also be a solution of the point charge-conducting plane problem, at least in the half space $z > 0$.

From the potential, we can also find the electric field from $\mathbf{E} = -\nabla V$. We can also find the surface charge density on the plane. At the surface of a conductor, we know that the charge density is given by the normal derivative of the potential at the surface:

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n} \tag{3}$$

In this case, the normal direction is in the z direction, so we get

$$\begin{aligned} \sigma &= -\epsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0} && (4) \\ &= \frac{-qd}{2\pi \left(x^2 + y^2 + d^2\right)^{3/2}} && (5) \end{aligned}$$

We can also calculate the total induced charge q_i by integrating this over the entire plane. This is easier to do in polar coordinates, where $r^2 = x^2 + y^2$ and the increment of area is $r\ dr\ d\theta$. We integrate σ to get

$$\begin{aligned} q_i &= \frac{-2\pi qd}{2\pi} \int_0^\infty \frac{r dr}{\left(r^2 + d^2\right)^{3/2}} && (6) \\ &= -q && (7) \end{aligned}$$

That is, the total induced charge is equal to the image charge.

Since the force is calculated from the electric field, and the field is calculated from the potential, the force between the point charge and the plane must be equal to the force between the point charge and image charge. That is

$$F = -\frac{q^2}{4\pi\epsilon_0 (4d^2)} \tag{8}$$

The work done to set up the configuration isn't quite as straightforward, since we can't calculate it directly from the two point charges. We can, however, work it out from the integral

$$W = \int \mathbf{F} \cdot d\mathbf{l} \tag{9}$$

We can choose a path from infinity on the z axis up to the location of the point charge at $z = d$. The force used in calculating the work done is the negative of the force between the charge and plane, since we're opposing the force, so we get

$$W = \int_{\infty}^{d} \frac{q^2 dz}{4\pi\epsilon_0 (4z^2)} \quad (10)$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d} \quad (11)$$

Notice this is half what we would get for the two point charges on their own, where $W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{2d}$. This isn't the same problem as that of finding the work done in assembling two point charges, since as we bring in one point charge, the location of the image charge changes to preserve the zero potential in the conducting plane. Another way of looking at it is that although we do work on the point charge in bringing it in from infinity, the induced charge in the plane is distributed by moving charge around a location where the potential is constant. Since the potential is constant, the field within the plane is zero, so no work is actually done to rearrange the induced charge. This does seem to be a bit of a fudge to me, though, since if there were really no transverse fields operating within the conducting plane, the charges within the plane wouldn't move at all, so I suspect there is *some* work done in the process that the theory at this level simply ignores. Comments welcome.

By using the superposition principle, we can apply the method of images to any number of point charges above the conducting plane. For example, suppose we have a grounded conducting plane in the xy plane, and a charge of $-2q$ is placed on the z axis at $z = d$, and a second charge of $+q$ is placed at $z = 3d$. We can use the method of images combined with the superposition principle to find the force on the charge of $+q$.

The $+q$ charge has an image of $-q$ at $z = -3d$ and the $-2q$ charge has an image of $+2q$ at $z = -d$.

Since the method of images allows us to find the potential from the image charges, the electric field (which is the negative gradient of the potential) must be the same in the image and original problems. Since the force is calculated from the field, it too is the same in the two problems.

Therefore, the force on $+q$ is

$$\mathbf{F} = \frac{q^2}{4\pi\epsilon_0}\left[-\frac{2}{4d^2}+\frac{2}{16d^2}-\frac{1}{36d^2}\right]\hat{\mathbf{z}} \tag{12}$$

$$= \frac{q^2}{4\pi\epsilon_0 d^2}\left[-\frac{1}{2}+\frac{1}{8}-\frac{1}{36}\right]\hat{\mathbf{z}} \tag{13}$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{29q^2}{72d^2}\hat{\mathbf{z}} \tag{14}$$

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