METHOD OF IMAGES - POINT CHARGE AND SPHERE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 3.7.

A special case in which the method of images works is that of a point charge and a grounded, conducting sphere. We'll take the sphere with its centre at the origin and give it a radius R. The point charge q is on the z axis at location $z=a$, where $a>R$ (so it's outside the sphere). Now we know that the potential everywhere on the sphere is zero (since it's a grounded conductor), so as the point charge is brought in, charge moves around on the conductor to keep the potential at that value. The problem is to find the potential everywhere outside the sphere.

Someone at some point noticed that if you replace the sphere by an image charge of $q'=-Rq/a$ at a location of $z=R^2/a$ (since $a>R$ this puts the image charge inside the sphere), then the potential on the sphere is still zero. This is most easily seen if we write the potential due to the two point charges in spherical coordinates. At a location $\mathbf{r}$ the potential due to the original charge is

$$V_q=\frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{r^2+a^2-2ar\cos\theta}} \tag{1}$$

where we've used the cosine rule to get the distance from q to the point $\mathbf{r}$. The angle θ is, as usual, the angle between the z axis and the vector $\mathbf{r}$.

By the same argument, the potential due to q' is

$$\begin{aligned}
V_{q'} &= \frac{1}{4\pi\epsilon_0}\frac{q'}{\sqrt{r^2+\left(R^2/a\right)^2-2r\left(R^2/a\right)\cos\theta}} && (2)\\
&= -\frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{\left(ar/R\right)^2+R^2-2ra\cos\theta}} && (3)
\end{aligned}$$

If we look at the surface of the sphere, then $r=R$ and we get $V_q=-V_{q'}$ so the total potential on the surface of the sphere is zero. As you might expect, this is a very special case, and it's highly unusual to find problems in which the method of images works this well.

However, having found the image charge, the net potential is just the sum of the two above, so we get

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}} - \frac{1}{\sqrt{(ar/R)^2+R^2-2ra\cos\theta}}\right] \tag{4}$$

From here, we can work out the induced surface charge on the sphere. For a conductor, we have for the surface charge density σ:

$$\sigma = -\epsilon_0\frac{\partial V}{\partial n} \tag{5}$$

That is, we need the derivative of the potential normal to the surface. In this case, the normal direction is in the direction of increasing **r**, so we just take the derivative with respect to r. This gives

$$\sigma = \frac{q}{4\pi}\left[\frac{r-a\cos\theta}{\left(r^2+a^2-2ra\cos\theta\right)^{3/2}} - \frac{ra^2/R^2-a\cos\theta}{\left((ar/R)^2+R^2-2ra\cos\theta\right)^{3/2}}\right]_{r=R} \tag{6}$$

$$= -\frac{q}{4\pi}\frac{a^2-R^2}{R\left(R^2+a^2-2Ra\cos\theta\right)^{3/2}} \tag{7}$$

The induced charge is negative, since q itself is positive. We can find the total induced charge on the sphere by integrating σ over the surface area.

$$q_i = -\frac{2\pi q}{4\pi}\int_0^\pi \frac{\left(a^2-R^2\right)R^2\sin\theta}{R\left(R^2+a^2-2Ra\cos\theta\right)^{3/2}}d\theta \tag{8}$$

$$= -\frac{qR}{a} \tag{9}$$

That is, the total induced charge is equal to the point image charge.

Finally, we can work out the energy of the configuration. The easiest way to do that is to recognize that the electric field and thus the force *outside* the sphere are the same in both configurations of the problem (the original one with the grounded sphere, and the image version with the two point charges). For a point charge at location $z_q(=a)$ and image charge at location $z = R^2/a$ the force between the charges is

$$F = \frac{qq'}{4\pi\epsilon_0}\frac{1}{\left(z_q - R^2/z_q\right)^2} \tag{10}$$

$$= -\frac{q^2}{4\pi\epsilon_0}\frac{R}{z_q\left(z_q - R^2/z_q\right)^2} \tag{11}$$

$$= -\frac{q^2R}{4\pi\epsilon_0}\frac{z_q}{\left(z_q^2 - R^2\right)^2} \tag{12}$$

The energy is the work done in bringing the charge q from infinity up to location $z_q = a$. Thus we need to integrate $\mathbf{F}\cdot d\mathbf{l}$ over a path between these two points. The easiest path to take is along the z axis. We get (again, we take the negative of the force between the charges since we're opposing this force in doing the work):

$$W = \frac{q^2R}{4\pi\epsilon_0}\int_\infty^a \frac{z_q}{\left(z_q^2 - R^2\right)^2}dz_q \tag{13}$$

$$= -\frac{q^2R}{8\pi\epsilon_0\left(a^2 - R^2\right)} \tag{14}$$

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