# METHOD OF IMAGES: TWO CONDUCTING PLANES

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Sec 3.2, Problem 3.10.

Another example of the method of images is the problem of a point charge $+q$ located at point $(x,y)=(a,b)$ in the first quadrant (at $z=0$), between two conducting planes that cover the $xz$ and $yz$ planes, thus these two conducting planes meet at right angles.

Following the procedure for the simpler problem of an image charge next to a single conducting plane, we can first place images of $-q$ at locations $(x,y)=(-a,b)$ and $(a,-b)$. If we stopped there, the potential would be

$$V=\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x-a)^2+(y-b)^2+z^2}}-\frac{1}{\sqrt{(x+a)^2+(y-b)^2+z^2}}-\frac{1}{\sqrt{(x-a)^2+(y+b)^2+z^2}}\right] \tag{1}$$

This clearly isn't zero on either of the conducting planes, so we need to add another image. If we try an image of $+q$ at $(x,y)=(-a,-b)$ then the potential is

$$V=\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x-a)^2+(y-b)^2+z^2}}-\frac{1}{\sqrt{(x+a)^2+(y-b)^2+z^2}}-\frac{1}{\sqrt{(x-a)^2+(y+b)^2+z^2}}+\frac{}{\sqrt{}}\right. \tag{2}$$

Now $V=0$ on both the planes $x=0$ and $y=0$ as can be seen by direct substitution.

The force on the original charge can be found from the force from the 3 images:

$$\mathbf{F}=\frac{q^2}{4\pi\varepsilon_0}\left(-\frac{1}{4a^2}\hat{\mathbf{x}}-\frac{1}{4b^2}\hat{\mathbf{y}}+\frac{1}{4\left(a^2+b^2\right)^{3/2}}\left(a\hat{\mathbf{x}}+b\hat{\mathbf{y}}\right)\right) \tag{3}$$

The third term is the force between the actual charge and the image at $(x,y)=(-a,-b)$. The magnitude of this force is $q^2/\left[4\pi\varepsilon_0\left(4a^2+4b^2\right)\right]$ and we've resolved this along the two coordinate axes.

The work required to bring in $q$ from infinity can be found from the general formula for work applied to the images:

$$W=\frac{1}{2}\sum_{i=1}^{n}q_{i}V(\mathbf{r}_{i}) \tag{4}$$

where $V(\mathbf{r}_i)$ is the potential due to all the charges in the collection except $q_i$. However, this work assumes that we're looking at all space, whereas the conducting planes cut the space under consideration down to a quarter of all space, so we need to divide the result by 4.

Plugging in the values we get

$$\begin{aligned}
V(a,b) &= \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{2a}-\frac{1}{2b}+\frac{1}{2\sqrt{a^2+b^2}}\right] && (5)\\
V(-a,b) &= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{2a}+\frac{1}{2b}-\frac{1}{2\sqrt{a^2+b^2}}\right] && (6)\\
V(-a,-b) &= \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{2a}-\frac{1}{2b}+\frac{1}{2\sqrt{a^2+b^2}}\right] && (7)\\
V(a,-b) &= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{2a}+\frac{1}{2b}-\frac{1}{2\sqrt{a^2+b^2}}\right] && (8)
\end{aligned}$$

Thus

$$\begin{aligned}
W &= \frac{1}{4}\frac{1}{2}\frac{q^2}{4\pi\varepsilon_0}4\left[-\frac{1}{2a}-\frac{1}{2b}+\frac{1}{2\sqrt{a^2+b^2}}\right] && (9)\\
&= \frac{q^2}{8\pi\varepsilon_0}\left[-\frac{1}{2a}-\frac{1}{2b}+\frac{1}{2\sqrt{a^2+b^2}}\right] && (10)
\end{aligned}$$

This technique can actually be applied to a configuration where we have two conducting planes meeting at other angles. Since the image charges have to cancel in pairs, we can derive a formula for the case where we can divide up space into an even number $n$ of sectors. We've just seen how it works for $n=4$, but for general even $n$ the argument would go like this.

Suppose the sector bounded by the planes has one plane at $y=0$ (that is, the $xz$ plane) and the other at an angle of $2\pi/n$. Let's place the test charge at a location $(x,y)=d(\cos\alpha,\sin\alpha)$ where $\alpha$ is the angle the radius vector to the charge makes with the plane $y=0$ and $d$ is the distance of the charge from the origin. We can then place the first image by drawing a line from the charge perpendicular to the plane at angle $2\pi/n$ and extending it an equal distance on the other side. This image will have charge $-q$ and be at an angle of $2\times\frac{2\pi}{n}-\alpha=\frac{4\pi}{n}-\alpha$. The next image is found by drawing a line from the first image perpendicular to the plane at angle $4\pi/n$ and extending it an equal distance on the other side. This charge is $+q$ and is at angle

$\frac{4\pi}{n}+\alpha$. We continue in this fashion until we arrive at a charge of $-q$ at angle $2\pi-\alpha$, which is the image of the original charge in the $y=0$ plane. This will give a total of $n$ charges (one is the test charge and the other $n-1$ are the images). The potential is then

$$\frac{4\pi\varepsilon_0}{q}V=\sum_{m=0}^{\frac{n}{2}-1}\frac{1}{\sqrt{\left(x-d\cos\left(\frac{4\pi m}{n}+\alpha\right)\right)^2+\left(y-d\sin\left(\frac{4\pi m}{n}+\alpha\right)\right)^2+z^2}}- \tag{11}$$

$$\sum_{m=1}^{\frac{n}{2}}\frac{1}{\sqrt{\left(x-d\cos\left(\frac{4\pi m}{n}-\alpha\right)\right)^2+\left(y-d\sin\left(\frac{4\pi m}{n}-\alpha\right)\right)^2+z^2}} \tag{12}$$