METHOD OF IMAGES: TWO CONDUCTING PLANES

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Sec 3.2, Problem 3.10.

Another example of the method of images is the problem of a point charge $+q$ located at point $(x,y)=(a,b)$ in the first quadrant (at $z=0$), between two conducting planes that cover the xz and yz planes, thus these two conducting planes meet at right angles.

Following the procedure for the simpler problem of an image charge next to a single conducting plane, we can first place images of $-q$ at locations $(x,y)=(-a,b)$ and $(a,-b)$. If we stopped there, the potential would be

(1)
$$V=\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x-a)^2+(y-b)^2+z^2}}-\frac{1}{\sqrt{(x+a)^2+(y-b)^2+z^2}}-\frac{1}{\sqrt{(x-a)^2+(y+b)^2+z^2}}\right]$$

This clearly isn't zero on either of the conducting planes, so we need to add another image. If we try an image of $+q$ at $(x,y)=(-a,-b)$ then the potential is

(2)
$$V=\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x-a)^2+(y-b)^2+z^2}}-\frac{1}{\sqrt{(x+a)^2+(y-b)^2+z^2}}-\frac{1}{\sqrt{(x-a)^2+(y+b)^2+z^2}}+\right.$$

Now $V=0$ on both the planes $x=0$ and $y=0$ as can be seen by direct substitution.

The force on the original charge can be found from the force from the 3 images:

$$\mathbf{F}=\frac{q^2}{4\pi\varepsilon_0}\left(-\frac{1}{4a^2}\hat{\mathbf{x}}-\frac{1}{4b^2}\hat{\mathbf{y}}+\frac{1}{4\left(a^2+b^2\right)^{3/2}}\left(a\hat{\mathbf{x}}+b\hat{\mathbf{y}}\right)\right) \tag{3}$$

The third term is the force between the actual charge and the image at $(x,y)=(-a,-b)$. The magnitude of this force is $q^2/\left[4\pi\varepsilon_0\left(4a^2+4b^2\right)\right]$ and we've resolved this along the two coordinate axes.

The work required to bring in q from infinity can be found from the general formula for work applied to the images:

$$W=\frac{1}{2}\sum_{i=1}^{n}q_iV(\mathbf{r}_i) \tag{4}$$

where $V(\mathbf{r}_i)$ is the potential due to all the charges in the collection except q_i. However, this work assumes that we're looking at all space, whereas the conducting planes cut the space under consideration down to a quarter of all space, so we need to divide the result by 4.

Plugging in the values we get

$$V(a,b)=\frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{2a}-\frac{1}{2b}+\frac{1}{2\sqrt{a^2+b^2}}\right] \tag{5}$$

$$V(-a,b)=\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{2a}+\frac{1}{2b}-\frac{1}{2\sqrt{a^2+b^2}}\right] \tag{6}$$

$$V(-a,-b)=\frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{2a}-\frac{1}{2b}+\frac{1}{2\sqrt{a^2+b^2}}\right] \tag{7}$$

$$V(a,-b)=\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{2a}+\frac{1}{2b}-\frac{1}{2\sqrt{a^2+b^2}}\right] \tag{8}$$

Thus

$$W=\frac{1}{4}\frac{1}{2}\frac{q^2}{4\pi\varepsilon_0}4\left[-\frac{1}{2a}-\frac{1}{2b}+\frac{1}{2\sqrt{a^2+b^2}}\right] \tag{9}$$

$$=\frac{q^2}{8\pi\varepsilon_0}\left[-\frac{1}{2a}-\frac{1}{2b}+\frac{1}{2\sqrt{a^2+b^2}}\right] \tag{10}$$

This technique can actually be applied to a configuration where we have two conducting planes meeting at other angles. Since the image charges have to cancel in pairs, we can derive a formula for the case where we can divide up space into an even number n of sectors. We've just seen how it works for $n=4$, but for general even n the argument would go like this.

Suppose the sector bounded by the planes has one plane at $y=0$ (that is, the xz plane) and the other at an angle of $2\pi/n$. Let's place the test charge at a location $(x,y)=d(\cos\alpha,\sin\alpha)$ where α is the angle the radius vector to the charge makes with the plane $y=0$ and d is the distance of the charge from the origin. We can then place the first image by drawing a line from the charge perpendicular to the plane at angle $2\pi/n$ and extending it an equal distance on the other side. This image will have charge $-q$ and be at an angle of $2\times\frac{2\pi}{n}-\alpha=\frac{4\pi}{n}-\alpha$. The next image is found by drawing a line from the first image perpendicular to the plane at angle $4\pi/n$ and extending it an equal distance on the other side. This charge is $+q$ and is at angle $\frac{4\pi}{n}+\alpha$. We continue in this fashion until we arrive at a charge of $-q$ at angle $2\pi-\alpha$, which is the image of the original charge in the $y=0$ plane. This will give a total of n charges (one is the test charge and the other $n-1$ are the images). The potential is then

$$\frac{4\pi\varepsilon_0}{q}V=\sum_{m=0}^{\frac{n}{2}-1}\frac{1}{\sqrt{\left(x-d\cos\left(\frac{4\pi m}{n}+\alpha\right)\right)^2+\left(y-d\sin\left(\frac{4\pi m}{n}+\alpha\right)\right)^2+z^2}}- \tag{11}$$

$$\sum_{m=1}^{\frac{n}{2}}\frac{1}{\sqrt{\left(x-d\cos\left(\frac{4\pi m}{n}-\alpha\right)\right)^2+\left(y-d\sin\left(\frac{4\pi m}{n}-\alpha\right)\right)^2+z^2}} \tag{12}$$