

## METHOD OF IMAGES: TWO CONDUCTING PLANES

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Sec 3.2, Problem 3.10.

Another example of the method of images is the problem of a point charge  $+q$  located at point  $(x,y) = (a,b)$  in the first quadrant (at  $z = 0$ ), between two conducting planes that cover the  $xz$  and  $yz$  planes, thus these two conducting planes meet at right angles.

Following the procedure for the simpler problem of an image charge next to a single conducting plane, we can first place images of  $-q$  at locations  $(x,y) = (-a,b)$  and  $(a,-b)$ . If we stopped there, the potential would be

$$(0.1) \quad V = \frac{q}{4\pi\epsilon_0} \left[ \frac{1}{\sqrt{(x-a)^2 + (y-b)^2 + z^2}} - \frac{1}{\sqrt{(x+a)^2 + (y-b)^2 + z^2}} - \frac{1}{\sqrt{(x-a)^2 + (y+b)^2 + z^2}} \right]$$

This clearly isn't zero on either of the conducting planes, so we need to add another image. If we try an image of  $+q$  at  $(x,y) = (-a,-b)$  then the potential is

$$(0.2) \quad V = \frac{q}{4\pi\epsilon_0} \left[ \frac{1}{\sqrt{(x-a)^2 + (y-b)^2 + z^2}} - \frac{1}{\sqrt{(x+a)^2 + (y-b)^2 + z^2}} - \frac{1}{\sqrt{(x-a)^2 + (y+b)^2 + z^2}} + \frac{1}{\sqrt{(x+a)^2 + (y+b)^2 + z^2}} \right]$$

Now  $V = 0$  on both the planes  $x = 0$  and  $y = 0$  as can be seen by direct substitution.

The force on the original charge can be found from the force from the 3 images:

$$(0.3) \quad \mathbf{F} = \frac{q^2}{4\pi\epsilon_0} \left( -\frac{1}{4a^2}\hat{\mathbf{x}} - \frac{1}{4b^2}\hat{\mathbf{y}} + \frac{1}{4(a^2 + b^2)^{3/2}}(a\hat{\mathbf{x}} + b\hat{\mathbf{y}}) \right)$$

The third term is the force between the actual charge and the image at  $(x,y) = (-a,-b)$ . The magnitude of this force is  $q^2/[4\pi\epsilon_0(4a^2 + 4b^2)]$  and we've resolved this along the two coordinate axes.

The work required to bring in  $q$  from infinity can be found from the general formula for work applied to the images:

$$(0.4) \quad W = \frac{1}{2} \sum_{i=1}^n q_i V(\mathbf{r}_i)$$

where  $V(\mathbf{r}_i)$  is the potential due to all the charges in the collection except  $q_i$ . However, this work assumes that we're looking at all space, whereas the conducting planes cut the space under consideration down to a quarter of all space, so we need to divide the result by 4.

Plugging in the values we get

$$(0.5) \quad V(a, b) = \frac{q}{4\pi\epsilon_0} \left[ -\frac{1}{2a} - \frac{1}{2b} + \frac{1}{2\sqrt{a^2 + b^2}} \right]$$

$$(0.6) \quad V(-a, b) = \frac{q}{4\pi\epsilon_0} \left[ \frac{1}{2a} + \frac{1}{2b} - \frac{1}{2\sqrt{a^2 + b^2}} \right]$$

$$(0.7) \quad V(-a, -b) = \frac{q}{4\pi\epsilon_0} \left[ -\frac{1}{2a} - \frac{1}{2b} + \frac{1}{2\sqrt{a^2 + b^2}} \right]$$

$$(0.8) \quad V(a, -b) = \frac{q}{4\pi\epsilon_0} \left[ \frac{1}{2a} + \frac{1}{2b} - \frac{1}{2\sqrt{a^2 + b^2}} \right]$$

Thus

$$(0.9) \quad W = \frac{1}{4} \frac{1}{2} \frac{q^2}{4\pi\epsilon_0} 4 \left[ -\frac{1}{2a} - \frac{1}{2b} + \frac{1}{2\sqrt{a^2 + b^2}} \right]$$

$$(0.10) \quad = \frac{q^2}{8\pi\epsilon_0} \left[ -\frac{1}{2a} - \frac{1}{2b} + \frac{1}{2\sqrt{a^2 + b^2}} \right]$$

This technique can actually be applied to a configuration where we have two conducting planes meeting at other angles. Since the image charges have to cancel in pairs, we can derive a formula for the case where we can divide up space into an even number  $n$  of sectors. We've just seen how it works for  $n = 4$ , but for general even  $n$  the argument would go like this.

Suppose the sector bounded by the planes has one plane at  $y = 0$  (that is, the  $xz$  plane) and the other at an angle of  $2\pi/n$ . Let's place the test charge at a location  $(x, y) = d(\cos \alpha, \sin \alpha)$  where  $\alpha$  is the angle the radius vector to the charge makes with the plane  $y = 0$  and  $d$  is the distance of the charge from the origin. We can then place the first image by drawing a line from the charge perpendicular to the plane at angle  $2\pi/n$  and extending it an equal distance on the other side. This image will have charge  $-q$  and be at an angle of  $2 \times \frac{2\pi}{n} - \alpha = \frac{4\pi}{n} - \alpha$ . The next image is found by drawing a line

from the first image perpendicular to the plane at angle  $4\pi/n$  and extending it an equal distance on the other side. This charge is  $+q$  and is at angle  $\frac{4\pi}{n} + \alpha$ . We continue in this fashion until we arrive at a charge of  $-q$  at angle  $2\pi - \alpha$ , which is the image of the original charge in the  $y = 0$  plane. This will give a total of  $n$  charges (one is the test charge and the other  $n - 1$  are the images). The potential is then

(0.11)

$$\frac{4\pi\epsilon_0}{q} V = \sum_{m=0}^{\frac{n}{2}-1} \frac{1}{\sqrt{(x - d \cos(\frac{4\pi m}{n} + \alpha))^2 + (y - d \sin(\frac{4\pi m}{n} + \alpha))^2 + z^2}} -$$

(0.12)

$$\sum_{m=1}^{\frac{n}{2}} \frac{1}{\sqrt{(x - d \cos(\frac{4\pi m}{n} - \alpha))^2 + (y - d \sin(\frac{4\pi m}{n} - \alpha))^2 + z^2}}$$