LAPLACE'S EQUATION - FOURIER SERIES EXAMPLES 1

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problems 3.12 - 3.13.

Here are a few examples of calculating the Fourier coefficients for some special cases.

Example 1. Consider the infinite slot problem with the boundary at $x=0$ consisting of a conducting strip with a constant potential of V_0. In this case we get

$$c_n = \frac{2V_0}{a}\int_0^a \sin\frac{n\pi y}{a}dy \tag{0.1}$$

$$= \frac{2V_0}{n\pi}(1-\cos n\pi) \tag{0.2}$$

The coefficients are thus zero for even n and $4V_0/\pi$ for odd n:

$$c_n = \begin{cases} 0 & n \text{ even} \\ \frac{4V_0}{n\pi} & n \text{ odd} \end{cases} \tag{0.3}$$

The potential is thus

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{e^{-n\pi x/a}}{n}\sin\frac{n\pi y}{a} \tag{0.4}$$

Example 2. Now suppose the boundary at $x=0$ consists of two conducting strips, insulated from each other and from the infinite sheets. The first strip, from $y=0$ to $y=a/2$ has a constant potential V_0 while the other strip, from $y=a/2$ to $y=a$ is held at potential $-V_0$.

Here, the coefficients c_n are given by

$$c_n = \frac{2V_0}{a}\left[\int_0^{a/2}\sin\frac{n\pi y}{a}dy - \int_{a/2}^{a}\sin\frac{n\pi y}{a}dy\right] \tag{0.5}$$

$$= \frac{2V_0}{n\pi}\left[1-2\cos\frac{n\pi}{2}+\cos n\pi\right] \tag{0.6}$$

If n is odd, this comes out to zero. If n is even, there are two cases. First, if $n = 2, 6, 10, \ldots$ the term in brackets is 4. If $n = 4, 8, 12, \ldots$ the term in brackets is zero. Thus we get

$$c_n = \begin{cases} 0 & n \text{ odd} \\ \frac{8V_0}{n\pi} & n = 2, 6, 10 \ldots \\ 0 & n = 4, 8, 12 \ldots \end{cases} \tag{0.7}$$

Thus the potential is

$$V(x, y) = \frac{8V_0}{\pi} \sum_{n=2,6,10\ldots}^{\infty} \frac{e^{-n\pi x/a}}{n} \sin \frac{n\pi y}{a} \tag{0.8}$$

$$= \frac{8V_0}{\pi} \sum_{n=0}^{\infty} \frac{e^{-(4n+2)\pi x/a}}{4n+2} \sin \frac{(4n+2)\pi y}{a} \tag{0.9}$$

where in the last line we've changed the index of summation since the non-zero terms in the first sum are just those with $n = 4m + 2$ starting at $m = 0$.

Example 3. The infinite slot with the strip at $x = 0$ held at potential V_0 has the solution

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{e^{-n\pi x/a}}{n} \sin \frac{n\pi y}{a} \tag{0.10}$$

For a conductor, the surface charge density can be found from the derivative taken normal to the surface:

$$\sigma = -\varepsilon_0 \left. \frac{\partial V}{\partial n} \right|_{x=0} \tag{0.11}$$

In this case, the normal to the surface is the x direction, so we get

$$\sigma = -\varepsilon_0 \frac{4V_0}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \left(-\frac{n\pi}{a} \right) \frac{e^{-n\pi x/a}}{n} \sin \left. \frac{n\pi y}{a} \right|_{x=0} \tag{0.12}$$

$$= \varepsilon_0 \frac{4V_0}{a} \sum_{n=1,3,5,\ldots}^{\infty} \sin \frac{n\pi y}{a} \tag{0.13}$$

This looks fine except for the problem that the series doesn't converge. Consider $y = a/2$. The series is then a sum of an alternating sequence of $+1$ and -1. The original series for the potential does converge at $x = 0$ due to the n in the denominator. Not sure what the solution to this is.