LAPLACE'S EQUATION - FOURIER SERIES EXAMPLES 3 - THREE DIMENSIONS

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 3.15.

The examples of solving Laplace's equation we've seen so far have all been essentially two-dimensional, so it's time to see a fully three dimensional problem.

We have a cube of side length a whose faces are conducting plates. The cube is placed with one corner at the origin, and the other corner at $(x,y,z)=(a,a,a)$, with its edges parallel to the coordinate axes. The face at $z=a$ is held at a constant potential of V_0 (it is insulated from the other faces). All other faces are grounded so their potential is $V=0$. We want to find the potential inside the cube.

We use the standard separation of variables technique we described earlier. We assume that

$$V(x,y,z)=X(x)Y(y)Z(z) \tag{1}$$

Plugging this into Laplace's equation and dividing through by V, we get three ordinary differential equations:

$$\frac{1}{X}\frac{d^2X}{dx^2} = C_1 \tag{2}$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = C_2 \tag{3}$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = C_3 \tag{4}$$

with the condition on the constants

$$C_1+C_2+C_3=0 \tag{5}$$

The boundary conditions are

$$V=\begin{cases}0 & x=0\\ 0 & x=a\\ 0 & y=0\\ 0 & y=a\\ 0 & z=0\\ V_0 & z=a\end{cases} \tag{6}$$

The boundary conditions involving x and y suggest that a sine solution is appropriate here, so we can try setting $C_1=-k^2<0, C_2=-l^2<0$ and $C_3=k^2+l^2>0$. Again, these choices are really inspired guesswork, and you should feel free to try changing the signs of the constants and discovering that the solution doesn't work out as nicely.

With these choices, we get, in a similar way to solving the two-dimensional problem:

$$X(x)=A\sin kx+B\cos kx \tag{7}$$

$$Y(y)=C\sin ly+D\cos ly \tag{8}$$

$$Z(z)=Ee^{\sqrt{k^2+l^2}z}+Fe^{-\sqrt{k^2+l^2}z} \tag{9}$$

The first boundary condition implies $B=0$, the third implies $D=0$ and the fifth implies $E=-F$. The second condition implies $k=n\pi/a$ and the fourth implies $l=m\pi/a$ for some positive integers n and m. Putting all this together, we get

$$X(x)=A_n\sin\frac{n\pi x}{a} \tag{10}$$

$$Y(y)=C_m\sin\frac{m\pi y}{a} \tag{11}$$

$$Z(z)=E_{nm}\left(e^{\sqrt{n^2+m^2}\pi z/a}-e^{-\sqrt{n^2+m^2}\pi z/a}\right) \tag{12}$$

$$=2E_{nm}\sinh\left(\frac{\sqrt{n^2+m^2}\pi z}{a}\right) \tag{13}$$

where we've added subscripts to the constants to indicate that the constants could be different for each choice of n and m.

This solution on its own clearly doesn't satisfy the sixth boundary condition for $z=a$, but as usual, we can get around this by creating a Fourier series from these individual solutions. The most general solution therefore has the form

$$V(x,y,z)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}c_{nm}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a}\sinh\left(\frac{\sqrt{n^2+m^2}\pi z}{a}\right) \tag{14}$$

where as usual we have merged the constants together: $c_{nm}\equiv 2A_nC_mE_{nm}$.

We can now apply the sixth boundary condition to obtain a formula for the coefficients c_{nm}. We set $z=a$ and then multiply both sides by $\sin\frac{n'\pi x}{a}\sin\frac{m'\pi y}{a}$ and integrate over x and y. Since $V=V_0$ at $z=a$, we get

(15)

$$V_0\int_0^a\sin\frac{n'\pi x}{a}dx\int_0^a\sin\frac{m'\pi y}{a}dy=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}c_{nm}\sinh\left(\sqrt{n^2+m^2}\pi\right)\int_0^a\sin\frac{n\pi x}{a}\sin\frac{n'\pi x}{a}dx\int_0^a\sin\frac{m\pi y}{a}$$

The left side is

$$V_0\int_0^a\sin\frac{n'\pi x}{a}dx\int_0^a\sin\frac{m'\pi y}{a}dy=\begin{cases}0 & n'\text{ or }m'\text{ even}\\ \frac{4a^2V_0}{n'm'\pi^2} & n'\text{ and }m'\text{ odd}\end{cases} \tag{16}$$

The terms on the right side are zero unless $n=n'$ and $m=m'$. The sum reduces to a single term, which is

$$c_{n'm'}\sinh\left(\sqrt{n'^2+m'^2}\pi\right)\frac{a^2}{4} \tag{17}$$

Dropping the primes to clean up the notation, we therefore get

$$c_{nm}=\begin{cases}0 & n\text{ or }m\text{ even}\\ \frac{16V_0}{nm\pi^2\sinh\left(\sqrt{n^2+m^2}\pi\right)} & n\text{ and }m\text{ odd}\end{cases} \tag{18}$$

The overall solution is therefore

$$V(x,y,z)=\frac{16V_0}{\pi^2}\sum_{n=1,3,5,\ldots}\sum_{m=1,3,5,\ldots}\frac{\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a}\sinh\left(\frac{\sqrt{n^2+m^2}\pi z}{a}\right)}{nm\sinh\left(\sqrt{n^2+m^2}\pi\right)} \tag{19}$$

Since this solution satisfies the boundary condition at $z=a$ we get the rather curious formula:

$$\sum_{n=1,3,5,\ldots}\sum_{m=1,3,5,\ldots}\frac{1}{nm}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a}=\frac{\pi^2}{16} \tag{20}$$

which is apparently true for all values of x and y such that $0 < x < a$ and $0 < y < a$.