LAPLACE'S EQUATION IN SPHERICAL COORDINATES: EXAMPLES 1

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problems 3.17, 3.18.

Example 1. A simple example of Laplace's equation in spherical coordinates is that of a spherical shell of radius R with a constant potential V_0 over its surface. We want to find the potential inside and outside the sphere.

The general solution in spherical coordinates was found in the last post:

$$V(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \tag{1}$$

Inside the sphere, all B_l are zero to prevent an infinity at the origin, so we get

$$V(r,\theta)=\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta) \tag{2}$$

At the spherical boundary, $r=R$ so we get

$$V_0=\sum_{l=0}^{\infty}A_l R^l P_l(\cos\theta) \tag{3}$$

The only Legendre polynomial that doesn't depend on θ is $P_0=1$, so it is only the $l=0$ term that is non-zero, and we get $A_0=V_0$, so inside the sphere, $V=V_0$ everywhere.

Outside the sphere, all $A_l=0$ to prevent the potential increasing to infinity for large r. Again, the potential must satisfy the boundary condition at $r=R$, and we get

$$V_0=\sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}P_l(\cos\theta) \tag{4}$$

As above, we discard all terms except for $l=0$, which gives $B_0=RV_0$, and

$$V=\frac{RV_0}{r} \tag{5}$$

Example 2. We saw that the coefficients A_l and B_l can be found by working out integrals, but in some special cases, it is easier to match up terms in the series on both sides of the equation. This happens if we can express V as a series of cosines (admittedly, this doesn't happen very often, but they are popular student exercises).

For example, suppose we have a spherical shell of radius R on which the potential is $V(\theta)=k\cos 3\theta$. Using some trig identities, we can convert the cosine term.

$$\begin{aligned}
\cos 3\theta &= \cos(2\theta+\theta) && (6)\\
&= \cos 2\theta\cos\theta-\sin 2\theta\sin\theta && (7)\\
&= (2\cos^2\theta-1)\cos\theta-2\sin^2\theta\cos\theta && (8)\\
&= 2\cos^3\theta-\cos\theta-2(1-\cos^2\theta)\cos\theta && (9)\\
&= 4\cos^3\theta-3\cos\theta && (10)
\end{aligned}$$

We can now apply this to the general solution 1. Inside the sphere, the B_l terms are all zero to prevent an infinity at the origin, so we get at the boundary:

$$V(R,\theta)=\sum_{l=0}^{\infty}A_lR^lP_l(\cos\theta) \tag{11}$$

$$k\left(4\cos^3\theta-3\cos\theta\right)=\sum_{l=0}^{\infty}A_lR^lP_l(\cos\theta) \tag{12}$$

Since the only terms appearing in the potential are of degree 1 and 3, only P_1 and P_3 appear in the series on the right. From tables of Legendre polynomials, we have

$$\begin{aligned}
P_1(\cos\theta) &= \cos\theta && (13)\\
P_3(\cos\theta) &= \frac{5}{2}\cos^3\theta-\frac{3}{2}\cos\theta && (14)
\end{aligned}$$

Matching up terms for the 3rd degree term, we get

$$4k = \frac{5}{2}A_3R^3 \tag{15}$$

$$A_3 = \frac{8k}{5R^3} \tag{16}$$

With this value of A_3, the $l = 3$ term in the series contributes a term $-\frac{12}{5}k\cos\theta$, so combining this with the $l = 1$ term and equating this to the degree 1 term on the LHS, we get

$$-3k = A_1R - \frac{12}{5}k \tag{17}$$

$$A_1 = -\frac{3k}{5R} \tag{18}$$

The potential inside the sphere is thus:

$$V_{in}(r,\theta) = \frac{k}{5}\left(-3\frac{r}{R}P_1(\cos\theta) + 8\frac{r^3}{R^3}P_3(\cos\theta)\right) \tag{19}$$

Outside the sphere, we can use the technique, except this time it is the A_l terms that are all zero to avoid an infinite potential for large r. We get, at the boundary:

$$V(R,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}P_l(\cos\theta) \tag{20}$$

$$k\left(4\cos^3\theta - 3\cos\theta\right) = \sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}P_l(\cos\theta) \tag{21}$$

For the 3rd degree term:

$$4k = \frac{5B_3}{2R^4} \tag{22}$$

$$B_3 = \frac{8kR^4}{5} \tag{23}$$

The $l = 3$ term contributes a term of $-\frac{12}{5}k\cos\theta$ as before, so combining this with the $l = 1$ term, we get

$$-3k = \frac{B_1}{R^2} - \frac{12}{5}k \tag{24}$$

$$B_1 = -\frac{3kR^2}{5} \tag{25}$$

The outside potential is

$$V_{out}(r,\theta)=\frac{k}{5}\left(-\frac{3R^{2}}{r^{2}}P_{1}(\cos\theta)+\frac{8R^{4}}{r^{4}}P_{3}(\cos\theta)\right) \tag{26}$$

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