LAPLACE'S EQUATION IN SPHERICAL COORDINATES: SURFACE CHARGE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problems 3.19, 3.18.

We found the general solution of Laplace's equation in spherical coordinates to be:

$$V(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \tag{1}$$

If the potential is specified as $V_0(\theta)$ on the surface of a sphere, and there are no charges inside or outside the sphere, we can use the relations between the potential and the surface charge density to find that charge density on the sphere. Since the electric field is discontinuous across a charge layer, we have

$$E_{\perp}^{above}-E_{\perp}^{below}=\frac{\sigma}{\varepsilon_0} \tag{2}$$

and the field is related to the potential by

$$\mathbf{E}=-\nabla V \tag{3}$$

we can use the radial symmetry of the problem to deduce that, at the surface of the sphere where $r=R$,

$$\left.\frac{\partial V}{\partial r}\right|_{out}-\left.\frac{\partial V}{\partial r}\right|_{in}=-\frac{\sigma}{\varepsilon_0} \tag{4}$$

For this problem, we know that inside the sphere, $B_l=0$ and outside the sphere, $A_l=0$. So we have

$$\left.\frac{\partial V}{\partial r}\right|_{in} = \sum_{l=0}^{\infty} lA_l r^{l-1} P_l(\cos\theta) \tag{5}$$

$$\left.\frac{\partial V}{\partial r}\right|_{out} = -\sum_{l=0}^{\infty} (l+1)\frac{B_l}{r^{l+2}} P_l(\cos\theta) \tag{6}$$

At the boundary, we must have

$$\sum_{l=0}^{\infty}(l+1)\frac{B_l}{R^{l+2}}P_l(\cos\theta) + \sum_{l=0}^{\infty} lA_l R^{l-1} P_l(\cos\theta) = \frac{\sigma(\theta)}{\varepsilon_0} \tag{7}$$

However, since the potential is also continuous across a charge layer, we must have

$$V_0(\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) \tag{8}$$

Since the Legendre polynomials are orthogonal, these two series must be equal term by term, so we get

$$A_l R^l = \frac{B_l}{R^{l+1}} \tag{9}$$

Multiplying 8 both sides by $P_m(\cos\theta)\sin\theta$ and integrating from 0 to π, we get, using the orthogonality of the polynomials

$$\int_0^{\pi} V_0(\theta) P_m(\cos\theta)\sin\theta d\theta = \frac{2}{2m+1} A_m R^m \tag{10}$$

$$\equiv C_m \tag{11}$$

Putting this back into 7 we get

$$\frac{\sigma(\theta)}{\varepsilon_0} = \sum_{l=0}^{\infty}\left[\frac{2l+1}{2R}(l+1) + \frac{2l+1}{2R} l\right] C_l P_l(\cos\theta) \tag{12}$$

$$= \frac{1}{2R}\sum_{l=0}^{\infty}(2l+1)^2 C_l P_l(\cos\theta) \tag{13}$$

We can apply this result to find the charge distribution for the case where $V_0 = k\cos 3\theta$. By using trig identities, we can expand the cosine and express V_0 in terms of the P_l:

$$V_0(\theta) = \frac{k}{5}\left(-3P_1(\cos\theta) + 8P_3(\cos\theta)\right) \tag{14}$$

From here, we see that σ has contributions only from $l=1$ and $l=3$, and we get from the above formula for the C_l coefficients:

$$C_1 = -\frac{3k}{15} \tag{15}$$

$$C_3 = \frac{8k}{35} \tag{16}$$

$$\sigma(\theta) = \frac{k\varepsilon_0}{R}\left[-\frac{9}{5}P_1(\cos\theta) + \frac{56}{5}P_3(\cos\theta)\right] \tag{17}$$

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