

LAPLACE'S EQUATION - CHARGED DISK

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 3.21

In Example 3 in an earlier post, we found that the potential of a uniformly charged disk of radius R can be worked out for points on the axis of the disk and is

$$(1) \quad V(z) = \frac{\sigma}{2\epsilon_0} \left(\sqrt{z^2 + R^2} - |z| \right)$$

In spherical coordinates, a point with coordinate z on the axis has coordinates $(r, 0)$ if $z > 0$ or (r, π) if $z < 0$. We can therefore write

$$(2) \quad V(r, 0) = V(r, \pi) = \frac{\sigma}{2\epsilon_0} \left(\sqrt{r^2 + R^2} - r \right)$$

This formula is valid only for points on the z axis, but we can use it, together with the general solution of Laplace's equation in terms of Legendre polynomials, to get an approximation for the potential off the z axis.

We consider first the case of $r > R$. In this case the most general solution of Laplace's equation in spherical coordinates is:

$$(3) \quad V(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

For $\theta = 0$ we get, since $P_l(0) = 1$ for all l :

$$(4) \quad V(r, 0) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}}$$

To find the coefficients B_l we need to expand 2. We can expand it in powers of (R/r) :

$$(5) \quad V(r, 0) = \frac{\sigma r}{2\epsilon_0} \left(\sqrt{1 + \frac{R^2}{r^2}} - 1 \right)$$

$$(6) \quad = \frac{\sigma}{2\epsilon_0} \left(\frac{R^2}{2r} - \frac{R^4}{8r^3} + \dots \right)$$

Comparing with 4, we get

$$(7) \quad B_0 = \frac{\sigma R^2}{4\epsilon_0}$$

$$(8) \quad B_1 = 0$$

$$(9) \quad B_2 = -\frac{\sigma R^4}{16\epsilon_0}$$

so the general formula for the potential off the axis is

$$(10) \quad V(r, \theta) = \frac{\sigma}{2\epsilon_0} \left[\frac{R^2}{2r} P_0(\cos \theta) - \frac{R^4}{8r^3} P_2(\cos \theta) + \dots \right]$$

Since the odd polynomial terms all vanish, and the even Legendre polynomials are even functions, this expansion is also valid for the region $z < 0$, where $\pi/2 < \theta \leq \pi$.

For the case $r < R$, we have

$$(11) \quad V(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta)$$

When $\theta = 0$, we get

$$(12) \quad V(r, 0) = \sum_{l=0}^{\infty} A_l r^l$$

We can now expand 2 in powers of r/R , and we get

$$(13) \quad V(r, 0) = \frac{\sigma}{2\epsilon_0} \left(R \sqrt{1 + \frac{r^2}{R^2}} - r \right)$$

$$(14) \quad = \frac{\sigma}{2\epsilon_0} \left(R - r + \frac{r^2}{2R} - \frac{r^4}{8R^3} + \dots \right)$$

Comparing this to 12, we get

$$(15) \quad A_0 = \frac{\sigma R}{2\epsilon_0}$$

$$(16) \quad A_1 = -\frac{\sigma}{2\epsilon_0}$$

$$(17) \quad A_2 = \frac{\sigma}{4\epsilon_0 R}$$

$$(18) \quad A_3 = 0$$

The off-axis expansion thus begins:

$$(19) \quad V(r, \theta) = \frac{\sigma}{2\epsilon_0} \left[RP_0(\cos \theta) - rP_1(\cos \theta) + \frac{r^2}{2R}P_2(\cos \theta) + \dots \right]$$

In this case, since there is a P_1 term, the same expansion isn't valid for $\theta = \pi$. However, because the odd polynomials are odd functions, for $\theta = \pi$ we have

$$(20) \quad V(r, \pi) = \sum_{l=0}^{\infty} (-1)^l A_l r^l$$

The only change we need to make in the expansion is thus to change the sign of A_1 so we get

$$(21) \quad V(r, \theta) = \frac{\sigma}{2\epsilon_0} \left[RP_0(\cos \theta) + rP_1(\cos \theta) + \frac{r^2}{2R}P_2(\cos \theta) + \dots \right]$$

All higher odd polynomials have $A_l = 0$ so this is the only change that is required in the entire expansion.

PINGBACKS

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