LAPLACE'S EQUATION - CHARGED DISK

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 3.21

In Example 3 in an earlier post, we found that the potential of a uniformly charged disk of radius R can be worked out for points on the axis of the disk and is

$$V(z)=\frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+R^2}-|z|\right) \tag{1}$$

In spherical coordinates, a point with coordinate z on the axis has coordinates $(r,0)$ if $z>0$ or (r,π) if $z<0$. We can therefore write

$$V(r,0)=V(r,\pi)=\frac{\sigma}{2\epsilon_0}\left(\sqrt{r^2+R^2}-r\right) \tag{2}$$

This formula is valid only for points on the z axis, but we can use it, together with the general solution of Laplace's equation in terms of Legendre polynomials, to get an approximation for the potential off the z axis.

We consider first the case of $r>R$. In this case the most general solution of Laplace's equation in spherical coordinates is:

$$V(r,\theta)=\sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta) \tag{3}$$

For $\theta=0$ we get, since $P_l(0)=1$ for all l:

$$V(r,0)=\sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}} \tag{4}$$

To find the coefficients B_l we need to expand 2. We can expand it in powers of (R/r):

$$V(r,0)=\frac{\sigma r}{2\epsilon_0}\left(\sqrt{1+\frac{R^2}{r^2}}-1\right) \tag{5}$$

$$=\frac{\sigma}{2\epsilon_0}\left(\frac{R^2}{2r}-\frac{R^4}{8r^3}+\ldots\right) \tag{6}$$

Comparing with 4, we get

$$B_0 = \frac{\sigma R^2}{4\epsilon_0} \tag{7}$$
$$B_1 = 0 \tag{8}$$
$$B_2 = -\frac{\sigma R^4}{16\epsilon_0} \tag{9}$$

so the general formula for the potential off the axis is

$$V(r,\theta) = \frac{\sigma}{2\epsilon_0}\left[\frac{R^2}{2r}P_0(\cos\theta) - \frac{R^4}{8r^3}P_2(\cos\theta) + \ldots\right] \tag{10}$$

Since the odd polynomial terms all vanish, and the even Legendre polynomials are even functions, this expansion is also valid for the region $z < 0$, where $\pi/2 < \theta \leq \pi$.

For the case $r < R$, we have

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \tag{11}$$

When $\theta = 0$, we get

$$V(r,0) = \sum_{l=0}^{\infty} A_l r^l \tag{12}$$

We can now expand 2 in powers of r/R, and we get

$$V(r,0) = \frac{\sigma}{2\epsilon_0}\left(R\sqrt{1+\frac{r^2}{R^2}} - r\right) \tag{13}$$
$$= \frac{\sigma}{2\epsilon_0}\left(R - r + \frac{r^2}{2R} - \frac{r^4}{8R^3} + \ldots\right) \tag{14}$$

Comparing this to 12, we get

$$A_0 = \frac{\sigma R}{2\epsilon_0} \tag{15}$$
$$A_1 = -\frac{\sigma}{2\epsilon_0} \tag{16}$$
$$A_2 = \frac{\sigma}{4\epsilon_0 R} \tag{17}$$
$$A_3 = 0 \tag{18}$$

The off-axis expansion thus begins:

$$V(r,\theta)=\frac{\sigma}{2\epsilon_0}\left[RP_0(\cos\theta)-rP_1(\cos\theta)+\frac{r^2}{2R}P_2(\cos\theta)+\ldots\right] \tag{19}$$

In this case, since there is a P_1 term, the same expansion isn't valid for $\theta=\pi$. However, because the odd polynomials are odd functions, for $\theta=\pi$ we have

$$V(r,\pi)=\sum_{l=0}^{\infty}(-1)^l A_l r^l \tag{20}$$

The only change we need to make in the expansion is thus to change the sign of A_1 so we get

$$V(r,\theta)=\frac{\sigma}{2\epsilon_0}\left[RP_0(\cos\theta)+rP_1(\cos\theta)+\frac{r^2}{2R}P_2(\cos\theta)+\ldots\right] \tag{21}$$

All higher odd polynomials have $A_l=0$ so this is the only change that is required in the entire expansion.

PINGBACKS

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