LAPLACE'S EQUATION - CHARGED DISK

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 3.21

In Example 3 in an earlier post, we found that the potential of a uniformly charged disk of radius R can be worked out for points on the axis of the disk and is

$$V(z)=\frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+R^2}-|z|\right) \tag{0.1}$$

In spherical coordinates, a point with coordinate z on the axis has coordinates $(r,0)$ if $z>0$ or (r,π) if $z<0$. We can therefore write

$$V(r,0)=V(r,\pi)=\frac{\sigma}{2\epsilon_0}\left(\sqrt{r^2+R^2}-r\right) \tag{0.2}$$

This formula is valid only for points on the z axis, but we can use it, together with the general solution of Laplace's equation in terms of Legendre polynomials, to get an approximation for the potential off the z axis.

We consider first the case of $r>R$. In this case the most general solution of Laplace's equation in spherical coordinates is:

$$V(r,\theta)=\sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta) \tag{0.3}$$

For $\theta=0$ we get, since $P_l(0)=1$ for all l:

$$V(r,0)=\sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}} \tag{0.4}$$

To find the coefficients B_l we need to expand 0.2. We can expand it in powers of (R/r):

$$V(r,0) = \frac{\sigma r}{2\varepsilon_0}\left(\sqrt{1+\frac{R^2}{r^2}}-1\right) \tag{0.5}$$

$$= \frac{\sigma}{2\varepsilon_0}\left(\frac{R^2}{2r}-\frac{R^4}{8r^3}+\ldots\right) \tag{0.6}$$

Comparing with 0.4, we get

$$B_0 = \frac{\sigma R^2}{4\varepsilon_0} \tag{0.7}$$

$$B_1 = 0 \tag{0.8}$$

$$B_2 = -\frac{\sigma R^4}{16\varepsilon_0} \tag{0.9}$$

so the general formula for the potential off the axis is

$$V(r,\theta) = \frac{\sigma}{2\varepsilon_0}\left[\frac{R^2}{2r}P_0(\cos\theta)-\frac{R^4}{8r^3}P_2(\cos\theta)+\ldots\right] \tag{0.10}$$

Since the odd polynomial terms all vanish, and the even Legendre polynomials are even functions, this expansion is also valid for the region $z<0$, where $\pi/2<\theta\leq\pi$.

For the case $r<R$, we have

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \tag{0.11}$$

When $\theta=0$, we get

$$V(r,0) = \sum_{l=0}^{\infty} A_l r^l \tag{0.12}$$

We can now expand 0.2 in powers of r/R, and we get

$$V(r,0) = \frac{\sigma}{2\varepsilon_0}\left(R\sqrt{1+\frac{r^2}{R^2}}-r\right) \tag{0.13}$$

$$= \frac{\sigma}{2\varepsilon_0}\left(R-r+\frac{r^2}{2R}-\frac{r^4}{8R^3}+\ldots\right) \tag{0.14}$$

Comparing this to 0.12, we get

$$A_0 = \frac{\sigma R}{2\varepsilon_0} \tag{0.15}$$

$$A_1 = -\frac{\sigma}{2\varepsilon_0} \tag{0.16}$$

$$A_2 = \frac{\sigma}{4\varepsilon_0 R} \tag{0.17}$$

$$A_3 = 0 \tag{0.18}$$

The off-axis expansion thus begins:

$$V(r,\theta) = \frac{\sigma}{2\varepsilon_0}\left[RP_0(\cos\theta) - rP_1(\cos\theta) + \frac{r^2}{2R}P_2(\cos\theta) + \ldots\right] \tag{0.19}$$

In this case, since there is a P_1 term, the same expansion isn't valid for $\theta = \pi$. However, because the odd polynomials are odd functions, for $\theta = \pi$ we have

$$V(r,\pi) = \sum_{l=0}^{\infty}(-1)^l A_l r^l \tag{0.20}$$

The only change we need to make in the expansion is thus to change the sign of A_1 so we get

$$V(r,\theta) = \frac{\sigma}{2\varepsilon_0}\left[RP_0(\cos\theta) + rP_1(\cos\theta) + \frac{r^2}{2R}P_2(\cos\theta) + \ldots\right] \tag{0.21}$$

All higher odd polynomials have $A_l = 0$ so this is the only change that is required in the entire expansion.

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