

LAPLACE'S EQUATION - CHARGED SPHERE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 3.22.

We've seen one example of the relation between surface charge and potential using the series solution to Laplace's equation. We consider a spherical shell of radius R and wish to find the potential from the surface charge on this shell. The general solution to Laplace's equation is

$$(1) \quad V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

Inside the sphere, to prevent an infinity at the origin, $B_l = 0$, while outside, to prevent infinities in the other direction, $A_l = 0$. In the earlier post, we found that:

$$(2) \quad \frac{\sigma(\theta)}{\epsilon_0} = \frac{1}{2R} \sum_{l=0}^{\infty} (2l+1)^2 C_l P_l(\cos \theta)$$

$$(3) \quad C_l \equiv \frac{2}{2l+1} A_l R^l$$

Simplifying, we get

$$(4) \quad \frac{\sigma(\theta)}{\epsilon_0} = \sum_{l=0}^{\infty} (2l+1) A_l R^{l-1} P_l(\cos \theta)$$

From here, we can use the orthogonality of the Legendre polynomials to get an expression for A_l . Multiply both sides by $P_m(\cos \theta) \sin \theta$ and integrate from 0 to π :

$$(5) \quad \frac{1}{\epsilon_0} \int_0^{\pi} \sigma(\theta) P_m(\cos \theta) \sin \theta d\theta = \sum_{l=0}^{\infty} (2l+1) A_l R^{l-1} \int_0^{\pi} P_m(\cos \theta) P_l(\cos \theta) \sin \theta d\theta$$

$$(6) \quad = \sum_{l=0}^{\infty} (2l+1) A_l R^{l-1} \frac{2}{2m+1} \delta_{lm}$$

$$(7) \quad = 2R^{m-1} A_m$$

So, reverting back to l as the index:

$$(8) \quad A_l = \frac{1}{2R^{l-1}\epsilon_0} \int_0^\pi \sigma(\theta) P_l(\cos \theta) \sin \theta d\theta$$

As an example, consider a spherical shell with a uniform charge density of $+\sigma_0$ in the region $0 \leq \theta < \pi/2$ and $-\sigma_0$ from $\pi/2 < \theta \leq \pi$. We can use 8 to work out the A_l explicitly. First, note that $\sin \theta$ is an even function over the interval $[0, \pi]$ (that is, $\sin(\pi/2 + x) = \sin(\pi/2 - x)$). Over the same interval, $P_l(\cos \theta)$ is even if l is even and odd if l is odd. The specified charge distribution is odd. (Another way is to convert the integral by the substitution $x = \cos \theta$ and then the functions are even or odd in the traditional way, with respect to the origin $x = 0$ as the mid-point.)

Therefore, only odd l terms will have a non-zero integral. For these terms, we need to work out

$$(9) \quad A_l = \frac{\sigma_0}{2R^{l-1}\epsilon_0} I_l$$

$$(10) \quad I_l \equiv \int_0^{\pi/2} P_l(\cos \theta) \sin \theta d\theta - \int_{\pi/2}^\pi P_l(\cos \theta) \sin \theta d\theta$$

We can look up tables of Legendre polynomials and work out these integrals manually (they all involve powers of the cosine multiplied by a single sine, so the integrals are fairly straightforward, but tedious). Or we can use software to do the integrals, and we get

$$(11) \quad I_0 = I_2 = I_4 = I_6 = \dots = 0$$

$$(12) \quad I_1 = 1$$

$$(13) \quad I_3 = -\frac{1}{4}$$

$$(14) \quad I_5 = \frac{1}{8}$$

The potential, up to $l = 6$, is therefore, for $r < R$:

$$(15) \quad V(r, \theta) = \frac{\sigma_0}{2\epsilon_0} \left[rP_1(\cos \theta) - \frac{r^3}{4R^2}P_3(\cos \theta) + \frac{r^5}{8R^4}P_5(\cos \theta) + \dots \right]$$

For $r > R$, we found in the earlier post that $B_l = A_l R^{2l+1}$ from the requirement that the potential be continuous at the boundary. Therefore

$$(16) \quad B_l = \frac{\sigma_0}{2\epsilon_0} R^{l+2} I_l$$

so

$$(17) \quad V(r, \theta) = \frac{\sigma_0}{2\epsilon_0} \left[\frac{R^3}{r^2} P_1(\cos \theta) - \frac{R^5}{4r^4} P_3(\cos \theta) + \frac{R^7}{8r^6} P_5(\cos \theta) + \dots \right]$$