LAPLACE'S EQUATION - CHARGED SPHERE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 3.22.

We've seen one example of the relation between surface charge and potential using the series solution to Laplace's equation. We consider a spherical shell of radius R and wish to find the potential from the surface charge on this shell. The general solution to Laplace's equation is

$$V(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \tag{1}$$

Inside the sphere, to prevent an infinity at the origin, $B_l=0$, while outside, to prevent infinities in the other direction, $A_l=0$. In the earlier post, we found that:

$$\frac{\sigma(\theta)}{\epsilon_0}=\frac{1}{2R}\sum_{l=0}^{\infty}(2l+1)^2 C_l P_l(\cos\theta) \tag{2}$$

$$C_l\equiv\frac{2}{2l+1}A_l R^l \tag{3}$$

Simplifying, we get

$$\frac{\sigma(\theta)}{\epsilon_0}=\sum_{l=0}^{\infty}(2l+1)A_l R^{l-1}P_l(\cos\theta) \tag{4}$$

From here, we can use the orthogonality of the Legendre polynomials to get an expression for A_l. Multiply both sides by $P_m(\cos\theta)\sin\theta$ and integrate from 0 to π:

$$\frac{1}{\epsilon_0}\int_0^{\pi}\sigma(\theta)P_m(\cos\theta)\sin\theta d\theta=\sum_{l=0}^{\infty}(2l+1)A_l R^{l-1}\int_0^{\pi}P_m(\cos\theta)P_l(\cos\theta)\sin\theta d\theta \tag{5}$$

$$=\sum_{l=0}^{\infty}(2l+1)A_l R^{l-1}\frac{2}{2m+1}\delta_{lm} \tag{6}$$

$$=2R^{m-1}A_m \tag{7}$$

So, reverting back to l as the index:

$$A_l = \frac{1}{2R^{l-1}\varepsilon_0}\int_0^\pi \sigma(\theta)P_l(\cos\theta)\sin\theta d\theta \tag{8}$$

As an example, consider a spherical shell with a uniform charge density of $+\sigma_0$ in the region $0 \le \theta < \pi/2$ and $-\sigma_0$ from $\pi/2 < \theta \le \pi$. We can use 8 to work out the A_l explicitly. First, note that $\sin\theta$ is an even function over the interval $[0,\pi]$ (that is, $\sin(\pi/2+x) = \sin(\pi/2-x)$). Over the same interval, $P_l(\cos\theta)$ is even if l is even and odd if l is odd. The specified charge distribution is odd. (Another way is to convert the integral by the substitution $x = \cos\theta$ and then the functions are even or odd in the traditional way, with respect to the origin $x = 0$ as the mid-point.)

Therefore, only odd l terms will have a non-zero integral. For these terms, we need to work out

$$A_l = \frac{\sigma_0}{2R^{l-1}\varepsilon_0} I_l \tag{9}$$

$$I_l \equiv \int_0^{\pi/2} P_l(\cos\theta)\sin\theta d\theta - \int_{\pi/2}^{\pi} P_l(\cos\theta)\sin\theta d\theta \tag{10}$$

We can look up tables of Legendre polynomials and work out these integrals manually (they all involve powers of the cosine multiplied by a single sine, so the integrals are fairly straightforward, but tedious). Or we can use software to do the integrals, and we get

$$I_0 = I_2 = I_4 = I_6 = \ldots = 0 \tag{11}$$

$$I_1 = 1 \tag{12}$$

$$I_3 = -\frac{1}{4} \tag{13}$$

$$I_5 = \frac{1}{8} \tag{14}$$

The potential, up to $l = 6$, is therefore, for $r < R$:

$$V(r,\theta) = \frac{\sigma_0}{2\varepsilon_0}\left[rP_1(\cos\theta) - \frac{r^3}{4R^2}P_3(\cos\theta) + \frac{r^5}{8R^4}P_5(\cos\theta) + \ldots\right] \tag{15}$$

For $r > R$, we found in the earlier post that $B_l = A_l R^{2l+1}$ from the requirement that the potential be continuous at the boundary. Therefore

$$B_l = \frac{\sigma_0}{2\varepsilon_0} R^{l+2} I_l \tag{16}$$

so

$$V(r,\theta)=\frac{\sigma_0}{2\varepsilon_0}\left[\frac{R^3}{r^2}P_1(\cos\theta)-\frac{R^5}{4r^4}P_3(\cos\theta)+\frac{R^7}{8r^6}P_5(\cos\theta)+\ldots\right] \quad (17)$$