LAPLACE'S EQUATION - INFINITE PIPE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 3.24.

Here's an example of using the solution to Laplace's equation in cylindrical coordinates. We have an infinite cylindrical conducting pipe of radius R lying with its axis on the z axis. A constant electric field $\mathbf{E}_0$ pointing in the $+x$ direction covers all space. Find the potential outside the pipe.

Since the pipe is a conductor, the potential is constant everywhere on it. In this case, the potential will not tend to zero at infinity, since there is an electric field everywhere, so we might as well choose the zero of V to be on the pipe. Far from the pipe, the potential must be

$$V_{far} = -E_0 x + C \tag{1}$$

We can choose $C = 0$ which means we are choosing $V = 0$ on the yz plane. The boundary conditions are therefore, in cylindrical coordinates:

$$V = \begin{cases} 0 & r = R \\ -E_0 r \cos\phi & r \to \infty \end{cases} \tag{2}$$

The solution of Laplace's equation in cylindrical coordinates is

$$V(r,\phi) = A\ln r + B + \sum_{n=1}^{\infty} r^n (A_n \sin n\phi + B_n \cos n\phi) + \sum_{n=-\infty}^{-1} r^n (C_n \sin n\phi + D_n \cos n\phi) \tag{3}$$

Since the sine is odd and the cosine is even, we can rewrite this as

$$V(r,\phi) = A\ln r + B + \sum_{n=1}^{\infty}\left(A_n r^n - \frac{C_n}{r^n}\right)\sin n\phi + \sum_{n=1}^{\infty}\left(B_n r^n + \frac{D_n}{r^n}\right)\cos n\phi \tag{4}$$

For $r \to \infty$, the terms with r^n in the denominator can be neglected, and we get

$$A_n = 0 \tag{5}$$
$$B_1 = -E_0 \tag{6}$$
$$B_n = 0 \quad (n \neq 1) \tag{7}$$
$$A = B = 0 \tag{8}$$

For $r = R$, we have

$$A_n R^n = \frac{C_n}{R^n} \tag{9}$$
$$B_n R^n = -\frac{D_n}{R^n} \tag{10}$$

Combining these two sets of results, we get

$$A_n = C_n = 0 \quad (all\ n) \tag{11}$$
$$B_1 = -E_0 \tag{12}$$
$$D_1 = E_0 R^2 \tag{13}$$
$$B_n = D_n = 0 \quad (n \neq 1) \tag{14}$$

Thus the final formula for the potential is

$$V(r,\phi) = E_0 \left(-r + \frac{R^2}{r}\right)\cos\phi \tag{15}$$

The induced charge density on the pipe can be found from the normal derivative of the potential at the surface of the conductor. In this case, the normal to the pipe is radially outwards, so we get

$$\sigma = -\varepsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=R} \tag{16}$$
$$= 2\varepsilon_0 E_0 \cos\phi \tag{17}$$

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