

MULTIPOLE EXPANSION IN ELECTROSTATICS

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Section 3.4.1 & Problem 3.26

For a given charge distribution, we can write down a *multipole expansion*, which gives the potential as a series in powers of $1/r$, where r is the distance from the origin to the observation point.

We know that the potential in general is

$$(1) \quad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}') \frac{d^3\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

In the integral, \mathbf{r}' is the position of charge element $\rho(\mathbf{r}')d^3\mathbf{r}'$. From the law of cosines

$$(2) \quad |\mathbf{r} - \mathbf{r}'| = \sqrt{r^2 + r'^2 - 2rr' \cos \theta'}$$

where θ' is the angle between \mathbf{r} and \mathbf{r}' . We can rewrite this as

$$(3) \quad \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \theta'}}$$

$$(4) \quad = \frac{1}{r} \frac{1}{\sqrt{1 + \frac{r'^2}{r^2} - 2\frac{r'}{r} \cos \theta'}}$$

From the theory of Legendre polynomials, it is known that the last factor in this expression is a *generating function* for the polynomials. That is, if we write the square root as an power series, we get

$$(5) \quad \frac{1}{\sqrt{1 + \frac{r'^2}{r^2} - 2\frac{r'}{r} \cos \theta'}} = \sum_{n=0}^{\infty} P_n(\cos \theta') \left(\frac{r'}{r}\right)^n$$

The coefficient of $\left(\frac{r'}{r}\right)^n$ in the series is the Legendre polynomial $P_n(\cos \theta')$. This can be verified for the first few terms by calculating the Taylor series expansion of the square root term about $r'/r = 0$. This is tedious to do by hand, but using Maple, we get, defining $s \equiv r'/r$:

$$(6) \quad \frac{1}{\sqrt{1+s^2-2s\cos\theta'}} = 1 + s\cos\theta' + s^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right) + s^3\left(\frac{5}{2}\cos^3\theta' - \frac{3}{2}\cos\theta'\right) + \dots$$

It is important to note that the angle θ' is equivalent to the angle θ in spherical coordinates *only* if the observation point \mathbf{r} lies on the z axis, since that is the only configuration where the angle between the observation vector and a charge element corresponds to the spherical coordinate angle θ . (A more general multipole expansion uses spherical harmonics rather than just Legendre polynomials, but that's a topic for a more advanced post.)

With this restriction, we can substitute the series expansion back into ?? to get

$$(7) \quad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r} \int \rho(\mathbf{r}') \sum_{n=0}^{\infty} P_n(\cos\theta') \left(\frac{r'}{r}\right)^n d^3\mathbf{r}'$$

$$(8) \quad = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int \rho(\mathbf{r}') P_n(\cos\theta') r'^n d^3\mathbf{r}'$$

The first few terms in this series have special names. The $n = 0$ term is

$$(9) \quad \frac{1}{4\pi\epsilon_0 r} \int \rho(\mathbf{r}') d^3\mathbf{r}' = \frac{Q}{4\pi\epsilon_0 r}$$

where Q is the total charge. This is called the *monopole* term, and shows that to a first approximation, the potential of any charge distribution is just the potential of a point charge with the same total charge.

The next term in the series is

$$(10) \quad \frac{1}{4\pi\epsilon_0 r^2} \int r' P_1(\cos\theta') \rho(\mathbf{r}') d^3\mathbf{r}' = \frac{1}{4\pi\epsilon_0 r^2} \int r' \cos\theta' \rho(\mathbf{r}') d^3\mathbf{r}'$$

This is called the *dipole* term.

For $n = 2$, we get the *quadrupole* term

$$(11) \quad \frac{1}{4\pi\epsilon_0 r^3} \int r'^2 P_2(\cos\theta') \rho(\mathbf{r}') d^3\mathbf{r}' = \frac{1}{4\pi\epsilon_0 r^3} \int r'^2 \left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right) \rho(\mathbf{r}') d^3\mathbf{r}'$$

Finally, for $n = 3$ we get the *octopole* term

$$(12) \quad \frac{1}{4\pi\epsilon_0 r^4} \int r'^3 P_3(\cos \theta') \rho(\mathbf{r}') d^3 \mathbf{r}' = \frac{1}{4\pi\epsilon_0 r^4} \int r'^3 \left(\frac{5}{2} \cos^3 \theta' - \frac{3}{2} \cos \theta' \right) \rho(\mathbf{r}') d^3 \mathbf{r}'$$

As an example, consider a solid sphere with a charge density

$$(13) \quad \rho(\mathbf{r}') = k \frac{R}{r'^2} (R - 2r') \sin \theta'$$

We can use the integrals above to find the first non-zero term in the series, and thus get an approximation for the potential. *Note that we can do this only for points on the z axis.*

By direct calculation, we have for the monopole term:

$$(15) \quad \frac{1}{4\pi\epsilon_0 r} \int \rho(\mathbf{r}') d^3 \mathbf{r}' = \frac{1}{4\pi\epsilon_0 r} \int_0^R \int_0^\pi \int_0^{2\pi} k \frac{R}{r'^2} (R - 2r') \sin \theta' r'^2 \sin \theta' d\phi' d\theta' dr' = 0$$

since the integral over r' gives zero. Thus the monopole term vanishes, as it always does if the total charge is zero.

For the dipole term, we get

$$(17) \quad \frac{1}{4\pi\epsilon_0 r^2} \int r' \cos \theta' \rho(\mathbf{r}') d^3 \mathbf{r}' = \frac{1}{4\pi\epsilon_0 r^2} \int_0^R \int_0^\pi \int_0^{2\pi} k \frac{R}{r'^2} (R - 2r') r'^3 \sin \theta' \cos \theta' \sin \theta' d\phi' d\theta' dr' = 0$$

This time, the integral over θ' gives zero, since the term $\cos \theta' \sin^2 \theta'$ is odd relative to the interval $[0, \pi]$.

For the quadrupole term

$$(18) \quad \frac{1}{4\pi\epsilon_0 r^3} \int r'^2 \left(\frac{3}{2} \cos^2 \theta' - \frac{1}{2} \right) \rho(\mathbf{r}') d^3 \mathbf{r}' = \frac{1}{4\pi\epsilon_0 r^3} \int_0^R \int_0^\pi \int_0^{2\pi} k \frac{R}{r'^2} (R - 2r') \sin \theta' \times r'^4 \left(\frac{3}{2} \cos^2 \theta' - \frac{1}{2} \right) \sin \theta' d\phi' d\theta' dr'$$

$$(19) \quad = \frac{1}{4\pi\epsilon_0 r^3} \left(\frac{\pi^2 k R^5}{48} \right)$$

$$(20) \quad = \frac{\pi k R^5}{192 \epsilon_0 r^3}$$

The octopole term comes out to zero, since the terms in θ' are again odd relative to the interval $[0, \pi]$.

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