MULTIPOLE EXPANSION IN ELECTROSTATICS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Section 3.4.1 & Problem 3.26

For a given charge distribution, we can write down a *multipole expansion*, which gives the potential as a series in powers of $1/r$, where r is the distance from the origin to the observation point.

We know that the potential in general is

$$V(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\int\rho(\mathbf{r}')\frac{d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \tag{0.1}$$

In the integral, $\mathbf{r}'$ is the position of charge element $\rho(\mathbf{r}')d^3\mathbf{r}'$. From the law of cosines

$$|\mathbf{r}-\mathbf{r}'|=\sqrt{r^2+r'^2-2rr'\cos\theta'} \tag{0.2}$$

where θ' is the angle between $\mathbf{r}$ and $\mathbf{r}'$. We can rewrite this as

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta'}} \tag{0.3}$$

$$=\frac{1}{r}\frac{1}{\sqrt{1+\frac{r'^2}{r^2}-2\frac{r'}{r}\cos\theta'}} \tag{0.4}$$

From the theory of Legendre polynomials, it is known that the last factor in this expression is a *generating function* for the polynomials. That is, if we write the square root as an power series, we get

$$\frac{1}{\sqrt{1+\frac{r'^2}{r^2}-2\frac{r'}{r}\cos\theta'}}=\sum_{n=0}^{\infty}P_n(\cos\theta')\left(\frac{r'}{r}\right)^n \tag{0.5}$$

The coefficient of $\left(\frac{r'}{r}\right)^n$ in the series is the Legendre polynomial $P_n(\cos\theta')$. This can be verified for the first few terms by calculating the Taylor series expansion of the square root term about $r'/r=0$. This is tedious to do by hand, but using Maple, we get, defining $s\equiv r'/r$:

(0.6)
$$\frac{1}{\sqrt{1+s^2-2s\cos\theta'}} = 1+s\cos\theta'+s^2\left(\frac{3}{2}\cos^2\theta'-\frac{1}{2}\right)+s^3\left(\frac{5}{2}\cos^3\theta'-\frac{3}{2}\cos\theta'\right)+\ldots$$

It is important to note that the angle θ' is equivalent to the angle θ in spherical coordinates *only* if the observation point $\mathbf{r}$ lies on the z axis, since that is the only configuration where the angle between the observation vector and a charge element corresponds to the spherical coordinate angle θ. (A more general multipole expansion uses spherical harmonics rather than just Legendre polynomials, but that's a topic for a more advanced post.)

With this restriction, we can substitute the series expansion back into **??** to get

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0 r}\int\rho(\mathbf{r}')\sum_{n=0}^{\infty}P_n(\cos\theta')\left(\frac{r'}{r}\right)^n d^3\mathbf{r}' \tag{0.7}$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int\rho(\mathbf{r}')P_n(\cos\theta')r'^n d^3\mathbf{r}' \tag{0.8}$$

The first few terms in this series have special names. The $n=0$ term is

$$\frac{1}{4\pi\varepsilon_0 r}\int\rho(\mathbf{r}')d^3\mathbf{r}' = \frac{Q}{4\pi\varepsilon_0 r} \tag{0.9}$$

where Q is the total charge. This is called the *monopole* term, and shows that to a first approximation, the potential of any charge distribution is just the potential of a point charge with the same total charge.

The next term in the series is

$$\frac{1}{4\pi\varepsilon_0 r^2}\int r'P_1(\cos\theta')\rho(\mathbf{r}')d^3\mathbf{r}' = \frac{1}{4\pi\varepsilon_0 r^2}\int r'\cos\theta'\rho(\mathbf{r}')d^3\mathbf{r}' \tag{0.10}$$

This is called the *dipole* term.

For $n=2$, we get the *quadrupole* term

(0.11)
$$\frac{1}{4\pi\varepsilon_0 r^3}\int r'^2P_2(\cos\theta')\rho(\mathbf{r}')d^3\mathbf{r}' = \frac{1}{4\pi\varepsilon_0 r^3}\int r'^2\left(\frac{3}{2}\cos^2\theta'-\frac{1}{2}\right)\rho(\mathbf{r}')d^3\mathbf{r}'$$

Finally, for $n=3$ we get the *octopole* term

$$\frac{1}{4\pi\varepsilon_0 r^4}\int r'^3 P_3(\cos\theta')\rho(\mathbf{r}')d^3\mathbf{r}' = \frac{1}{4\pi\varepsilon_0 r^4}\int r'^3\left(\frac{5}{2}\cos^3\theta' - \frac{3}{2}\cos\theta'\right)\rho(\mathbf{r}')d^3\mathbf{r}' \tag{0.12}$$

As an example, consider a solid sphere with a charge density

$$\rho(\mathbf{r}') = k\frac{R}{r'^2}(R-2r')\sin\theta' \tag{0.13}$$

We can use the integrals above to find the first non-zero term in the series, and thus get an approximation for the potential. *Note that we can do this only for points on the z axis.*

By direct calculation, we have for the monopole term:

$$\frac{1}{4\pi\varepsilon_0 r}\int \rho(\mathbf{r}')d^3\mathbf{r}' = \frac{1}{4\pi\varepsilon_0 r}\int_0^R\int_0^\pi\int_0^{2\pi} k\frac{R}{r'^2}(R-2r')\sin\theta' r'^2\sin\theta' d\phi' d\theta' dr' \tag{0.14}$$

$$= 0 \tag{0.15}$$

since the integral over r' gives zero. Thus the monopole term vanishes, as it always does if the total charge is zero.

For the dipole term, we get

$$\frac{1}{4\pi\varepsilon_0 r^2}\int r'\cos\theta'\rho(\mathbf{r}')d^3\mathbf{r}' = \frac{1}{4\pi\varepsilon_0 r^2}\int_0^R\int_0^\pi\int_0^{2\pi} k\frac{R}{r'^2}(R-2r')r'^3\sin\theta'\cos\theta'\sin\theta' d\phi' d\theta' dr' \tag{0.16}$$

$$= 0 \tag{0.17}$$

This time, the integral over θ' gives zero, since the term $\cos\theta'\sin^2\theta'$ is odd relative to the interval $[0,\pi]$.

For the quadrupole term

$$\frac{1}{4\pi\varepsilon_0 r^3}\int r'^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\mathbf{r}')d^3\mathbf{r}' = \frac{1}{4\pi\varepsilon_0 r^3}\int_0^R\int_0^\pi\int_0^{2\pi} k\frac{R}{r'^2}(R-2r')\sin\theta' \times$$

$$r'^4\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\sin\theta' d\phi' d\theta' dr' \tag{0.18}$$

$$= \frac{1}{4\pi\varepsilon_0 r^3}\left(\frac{\pi^2 kR^5}{48}\right) \tag{0.19}$$

$$= \frac{\pi kR^5}{192\varepsilon_0 r^3} \tag{0.20}$$

The octopole term comes out to zero, since the terms in θ' are again odd relative to the interval $[0,\pi]$.

PINGBACKS

Pingback: Dipoles
Pingback: Multipole expansion of linear charge
Pingback: Quadrupole moment
Pingback: Magnetic dipole