DIPOLE-DIPOLE INTERACTIONS

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Section 4.1.3; Problems 3.33, 4.5, 4.6.

We've seen that the electric field due to a dipole (aligned so its centre is at the origin and $\mathbf{p}$ points along the $+z$ axis) is (in spherical coordinates):

$$\mathbf{E} = \frac{p}{4\pi\varepsilon_0 r^3}\left[2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}\right] \tag{1}$$

We can express this in a coordinate-free form as follows. The angle θ is the angle between $\mathbf{p}$ and the unit vector $\hat{\mathbf{r}}$ in the radial direction, so $\mathbf{p}\cdot\hat{\mathbf{r}} = p\cos\theta$. The unit vector $\hat{\theta}$ makes an angle of $\theta + \pi/2$ with $\mathbf{p}$, so $\mathbf{p}\cdot\hat{\theta} = p\cos(\theta+\pi/2) = -p\sin\theta$. Therefore

$$\mathbf{p} = (\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} + (\mathbf{p}\cdot\hat{\theta})\hat{\theta} \tag{2}$$

$$= p\cos\theta\hat{\mathbf{r}} - p\sin\theta\hat{\theta} \tag{3}$$

So

$$\mathbf{E} = \frac{p}{4\pi\varepsilon_0 r^3}\left[2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}\right] \tag{4}$$

$$= \frac{1}{4\pi\varepsilon_0 r^3}\left[2(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p} + (\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\right] \tag{5}$$

$$= \frac{1}{4\pi\varepsilon_0 r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right] \tag{6}$$

For a physical dipole where charges $+q$ and $-q$ are separated by a vector distance $\mathbf{d}$, the net force in a uniform field $\mathbf{E}$ is zero, since the force on the positive charge cancels that on the negative one. However, there is a torque acting about the centre point. This torque $\mathbf{N}$ is

$$\mathbf{N} = \mathbf{r}_+ \times \mathbf{F}_+ + \mathbf{r}_- \times \mathbf{F}_- \tag{7}$$

$$= \frac{1}{2}\mathbf{d} \times (q\mathbf{E}) + \left[-\frac{1}{2}\mathbf{d} \times (-q\mathbf{E})\right] \tag{8}$$

$$= q\mathbf{d} \times \mathbf{E} \tag{9}$$

$$= \mathbf{p} \times \mathbf{E} \tag{10}$$

Although this derivation isn't technically valid for an ideal dipole, since the separation is zero there, we can work out a couple of problems assuming it's true.

Example 1. We have two perfect dipoles $\mathbf{p}_1$ and $\mathbf{p}_2$ separated by a distance r. Their alignment is such that $\mathbf{p}_1$ is perpendicular to the line separating them (pointing upwards) and $\mathbf{p}_2$ is parallel to the line separating them (pointing away from $\mathbf{p}_1$). From the formula above, the field $\mathbf{E}_{12}$ due to $\mathbf{p}_1$ at $\mathbf{p}_2$ is

$$\mathbf{E}_{12} = \frac{1}{4\pi\varepsilon_0 r^3}\left[3(\mathbf{p}_1 \cdot \hat{\mathbf{r}}_{12})\hat{\mathbf{r}}_{12} - \mathbf{p}_1\right] \tag{11}$$

$$= \frac{-1}{4\pi\varepsilon_0 r^3}\mathbf{p}_1 \tag{12}$$

since $\mathbf{p}_1 \perp \hat{\mathbf{r}}_{12}$. The torque on $\mathbf{p}_2$ is therefore

$$\mathbf{N}_2 = \frac{-1}{4\pi\varepsilon_0 r^3}\mathbf{p}_2 \times \mathbf{p}_1 \tag{13}$$

$$= \frac{1}{4\pi\varepsilon_0 r^3}\mathbf{p}_1 \times \mathbf{p}_2 \tag{14}$$

The field $\mathbf{E}_{21}$ due to $\mathbf{p}_2$ on $\mathbf{p}_1$ is

$$\mathbf{E}_{21} = \frac{1}{4\pi\varepsilon_0 r^3}\left[3(\mathbf{p}_2 \cdot \hat{\mathbf{r}}_{21})\hat{\mathbf{r}}_{21} - \mathbf{p}_2\right] \tag{15}$$

$$= \frac{1}{4\pi\varepsilon_0 r^3}\left[-3p_2\hat{\mathbf{r}}_{21} - \mathbf{p}_2\right] \tag{16}$$

$$= \frac{2}{4\pi\varepsilon_0 r^3}\mathbf{p}_2 \tag{17}$$

since $\mathbf{p}_2$ is anti-parallel to $\hat{\mathbf{r}}_{21}$. The torque on $\mathbf{p}_1$ is therefore

$$\mathbf{N}_1 = \frac{2}{4\pi\varepsilon_0 r^3}\mathbf{p}_1 \times \mathbf{p}_2 \tag{18}$$

This torque is in the same direction as the first one, and twice the magnitude. This appears to violate Newton's third law, in that the torque exerted by one dipole on the other is not equal and opposite to the inverse situation. The problem is because the two torques are calculated relative to different points, so we're not comparing like with like. See here for a resolution of this problem.

Example 2. A perfect dipole is situated a distance z above an infinite, grounded, conducting plane. The dipole moment $\mathbf{p}$ makes an angle θ with the perpendicular axis to the plane. What is the torque on the dipole?

Infinite grounded conducting planes make us think of the method of images. To see what the image of a dipole in a plane is, suppose it's a physical dipole with a finite separation between the two charges. If the positive charge is further away from the plane at a distance d_1, then in the image this charge will be a distance $-d_1$ and be negative. Similarly, the negative charge will have a positive image in the plane. The result is that dipole's image has the *same* z-component but opposite components in the x and y directions. That is, the angle between the dipole and its image is 2θ.

The field at the dipole from its image $\mathbf{p}'$, using the formula above and taking $\hat{\mathbf{r}}$ in the $+z$ direction is

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0 r^3}\left[3(\mathbf{p}'\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}'\right] \tag{19}$$

$$= \frac{1}{4\pi\varepsilon_0 r^3}\left[3p\cos\theta\hat{\mathbf{r}} - \mathbf{p}'\right] \tag{20}$$

The last line uses the fact that the magnitudes of the dipole and its image are the same.

The torque is then

$$\mathbf{N} = \mathbf{p}\times\mathbf{E} \tag{21}$$

$$= \frac{1}{4\pi\varepsilon_0 r^3}\left[3p\cos\theta\mathbf{p}\times\hat{\mathbf{r}} - \mathbf{p}\times\mathbf{p}'\right] \tag{22}$$

$$= \frac{p^2}{4\pi\varepsilon_0 r^3}\left(3\cos\theta\sin\theta - \sin 2\theta\right)\hat{\mathbf{x}} \tag{23}$$

$$= \frac{p^2}{4\pi\varepsilon_0 r^3}\left(\frac{1}{2}\sin 2\theta\right)\hat{\mathbf{x}} \tag{24}$$

We're assuming the coordinate system is aligned so that the x axis is perpendicular to $\mathbf{p}$ and $\hat{\mathbf{r}}$.

Thus the torque is zero when $\theta = 0$ or $\theta = \pi/2$, so the dipole feels no torque when its axis is either perpendicular or parallel to the plane. However, the position parallel to the plane is an unstable equilibrium, since the torque always tends to rotate the dipole so that it is perpendicular to the plane.

PINGBACKS

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