METHOD OF IMAGES - MOVING CHARGE AND PLANE

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 3.34.

As an example of the method of images, consider a point charge $+q$ of mass m that starts at rest a distance d from an infinite conducting plane. How long will it take for the charge to hit the plane?

We've seen that this setup can be replaced by an equal and opposite charge $-q$ a distance d on the other side of the plane. The force on the charge is therefore

$$F = -\frac{q^2}{4\pi\varepsilon_0(4d^2)} \tag{1}$$

If we let r be the distance of the charge from the plane, and let it be a function of time t, and use Newton's law we get (a dot above a variable indicates a time derivative; a double dot is the second derivative):

$$m\ddot{r} = -\frac{q^2}{4\pi\varepsilon_0(4r^2)} \tag{2}$$

$$\ddot{r} = -\frac{C}{r^2} \tag{3}$$

$$C \equiv \frac{q^2}{16\pi\varepsilon_0 m} \tag{4}$$

We therefore need to solve the differential equation

$$\ddot{r} = -\frac{C}{r^2} \tag{5}$$

subject to the initial conditions

$$r(0) = d \tag{6}$$

$$\dot{r}(0) = 0 \tag{7}$$

Solving the equation requires a trick, in that we multiply both sides by $\dot{r}$:

$$\dot{r}\ddot{r} = -C\frac{\dot{r}}{r^2} \tag{8}$$

We need to integrate this from $r = d$ to $r = 0$. We can integrate the LHS by parts to get (using the initial condition $\dot{r}(0) = 0$):

$$\int_0^t \dot{r}\ddot{r}dt' = \dot{r}^2\big|_0^t - \int_0^t \ddot{r}\dot{r}dt \tag{9}$$

$$\int_0^t \dot{r}\ddot{r}dt = \frac{1}{2}\dot{r}^2(t) \tag{10}$$

On the RHS, we get, also by integrating by parts and using $r(0) = d$:

$$-C\int_0^t \frac{\dot{r}}{r^2}dt = -C\left[\frac{1}{r(t)} - \frac{1}{d}\right] - 2C\int_0^t \frac{\dot{r}}{r^2}dt \tag{11}$$

$$C\int_0^t \frac{\dot{r}}{r^2}dt = -C\left[\frac{1}{r(t)} - \frac{1}{d}\right] \tag{12}$$

Combining the two sides, we get

$$\frac{1}{2}\dot{r}^2(t) = C\left[\frac{1}{r(t)} - \frac{1}{d}\right] \tag{13}$$

$$\dot{r}(t) = -\sqrt{2C}\left[\frac{1}{r(t)} - \frac{1}{d}\right]^{1/2} \tag{14}$$

$$= -\sqrt{2C}\sqrt{\frac{d-r}{rd}} \tag{15}$$

We've taken the negative square root on line 2 since r is decreasing with time. Note also that this equation satisfies the initial condition $\dot{r}(0) = 0$, since $r(0) = d$.

We can separate the variables and get

$$\int_d^0 \sqrt{\frac{rd}{d-r}}dr = -\sqrt{2C}\int_0^t dt' \tag{16}$$

The integral on the left can be done with software, and we get

$$-\frac{1}{2}d^{3/2}\pi = -\sqrt{2C}t \tag{17}$$

$$t = \frac{d^{3/2}\pi}{2\sqrt{2C}} \tag{18}$$

$$= \frac{d^{3/2}\pi 4\sqrt{\pi\varepsilon_0 m}}{2\sqrt{2}q} \tag{19}$$

$$= \frac{(\pi d)^{3/2}\sqrt{2\varepsilon_0 m}}{q} \tag{20}$$