METHOD OF IMAGES: POINT CHARGE AND TWO PLANES

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 3.35.

Given two parallel, grounded, infinite conducting planes a distance a apart, we place a charge $+q$ between the plates, a distance x from one of them. What is the force on the charge?

At first glance, this might seem easy, since you might think all we need to do is create an image charge behind each of the two planes. However, we need to maintain the potential on both planes at $V=0$, and introducing an image charge behind one plane messes up the potential on the other plane.

The only solution is to introduce an infinite sequence of images, where each image counters the one introduced just before it. If we imagine the two planes to be oriented vertically with our point of view looking at them edge on. Then with the charge at position x, we place the first image of $-q$ at distance x on the other side of the left plane (that is, a distance of $2x$ from the charge). This image is a distance $x+a$ from the right plane, so we need to place an image of $+q$ a distance $x+a$ behind the right plane. This image is a distance $(a-x)+x+a=2a$ from the charge. This latest image is now a distance of $x+2a$ from the left plane, so we need another image of $-q$ a distance of $x+2a$ behind the left plane (a distance of $2x+2a$ from the charge). The process continues like this, and repeats for the right plane, so we get a series of images as follows:

Images of $-q$ behind the left plane (to the left of the charge)

Distance from plane	Distance to left of charge
x	$2x$
$2a+x$	$2a+2x$
$4a+x$	$4a+2x$

Images of $+q$ behind the left plane:

Distance from plane	Distance to left of charge
$2a-x$	$2a$
$4a-x$	$4a$
$6a-x$	$6a$

Images of $-q$ behind the right plane (to the right of the charge)

Distance from plane	Distance to right of charge
$a-x$	$2a-2x$
$3a-x$	$4a-2x$
$5a-x$	$6a-2x$

Images of $+q$ behind the right plane.

Distance from plane	Distance to right of charge
$a+x$	$2a$
$3a+x$	$4a$
$5a+x$	$6a$

To work out the force, it is the distances of the images from the charge that we must use. We will count a force as negative if it pulls the charge to the left, and positive if it pulls it to the right. The forces from the positive images cancel out, and from the negative images we get:

$$F=\frac{q^2}{4\pi\varepsilon_0}\left[\frac{-1}{(2x)^2}-\frac{1}{(2a+2x)^2}-\frac{1}{(4a+2x)^2}-\ldots+\frac{1}{(2a-2x)^2}+\frac{1}{(4a-2x)^2}+\ldots\right] \tag{1}$$

I don't think this series has any closed form representation, so that's about as simple as we can get the final answer.

We can test it with a couple of special cases. If $a\to\infty$, then one of the plates is taken off to infinity and we're left with the simple case of a point charge a distance x from a single conducting plane. In that case, all the terms in the sum tend to zero except the first one, and we get

$$F\to-\frac{q^2}{4\pi\varepsilon_0(2x)^2} \tag{2}$$

In the case where $x=a/2$ so the charge is exactly halfway between the plates, the force should be zero by symmetry. Plugging this value into the force series above, we get

$$F(a/2)=\frac{q^2}{4\pi\varepsilon_0}\left[\frac{-1}{(a)^2}-\frac{1}{(3a)^2}-\frac{1}{(5a)^2}-\ldots+\frac{1}{(a)^2}+\frac{1}{(3a)^2}+\ldots\right] \tag{3}$$

We see that the terms all cancel in pairs, so the result appears to be correct.