LAPLACE'S EQUATION: CONDUCTING SPHERE AND SHELL OF CHARGE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 3.37.

Another example of using the series solution to Laplace's equation to find the potential using boundary conditions.

We have a conducting sphere of radius a maintained at a constant potential V_0. Concentric with this sphere is a larger spherical shell of radius b with a surface charge density of $\sigma(\theta) = k\cos\theta$. Find the potential in the two regions (i) $a \le r \le b$ and (ii) $r > b$.

The general solution in terms of Legendre polynomials is

$$V(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + \frac{B_l}{r^{l+1}}\right]P_l(\cos\theta) \tag{1}$$

In the inner region, we must use the most general form of the solution, since r tends neither to zero nor infinity. In the outer region, to keep the potential finite, we dispose of the terms in r^l, so we have

$$V_{out}(r,\theta) = \sum_{l=0}^{\infty}\frac{C_l}{r^{l+1}}P_l(\cos\theta) \tag{2}$$

Note that we must use a different set of coefficients since the potential will have a different functional form outside the shell.

We can now apply the various boundary conditions. First, at $r = a$ we must have $V = V_0$ so

$$\sum_{l=0}^{\infty}\left[A_l a^l + \frac{B_l}{a^{l+1}}\right]P_l(\cos\theta) = V_0 \tag{3}$$

From the orthogonality of the P_l, only the $l = 0$ term on the left is nonzero, so we get

$$A_0 + \frac{B_0}{a} = V_0 \tag{4}$$

$$B_l = -a^{2l+1}A_l \quad (l > 0) \tag{5}$$

Next, we look at $r=b$. Here, the potential must be continuous, so

$$\sum_{l=0}^{\infty}\left[A_l b^l+\frac{B_l}{b^{l+1}}\right]P_l(\cos\theta)=\sum_{l=0}^{\infty}\frac{C_l}{r^{l+1}}P_l(\cos\theta) \tag{6}$$

Equating coefficients of the P_l we get

$$A_l+\frac{B_l}{b^{l+1}}=\frac{C_l}{b^{l+1}} \tag{7}$$

Using the above relation between A_l and B_l we get, first for $l=0$:

$$A_0+\frac{B_0}{b}=\frac{C_0}{b} \tag{8}$$

and for $l>0$:

$$A_l\left(b^l-\frac{a^{2l+1}}{b^{l+1}}\right) = \frac{C_l}{b^{l+1}} \tag{9}$$

$$C_l = A_l\left(b^{2l+1}-a^{2l+1}\right) \tag{10}$$

Finally, we use the surface charge density on the shell. From our earlier example of this type, we get

$$\left.\frac{\partial V}{\partial r}\right|_{out}-\left.\frac{\partial V}{\partial r}\right|_{in}=-\frac{\sigma}{\varepsilon_0} \tag{11}$$

$$\sum_{l=0}^{\infty}\left[-(l+1)\frac{C_l}{b^{l+2}}-lA_l b^{l-1}+(l+1)\frac{B_l}{b^{l+2}}\right]P_l(\cos\theta)=-\frac{k}{\varepsilon_0}\cos\theta \tag{12}$$

Since $P_1(\cos\theta)=\cos\theta$, we can equate coefficients on both sides to get

$$-\frac{C_0}{b^2}+\frac{B_0}{b^2}=0 \tag{13}$$

$$-2\frac{C_1}{b^3}-A_1+2\frac{B_1}{b^3}=-\frac{k}{\varepsilon_0} \tag{14}$$

$$-(l+1)\frac{C_l}{b^{l+2}}-lA_l b^{l-1}+(l+1)\frac{B_l}{b^{l+2}}=0 \qquad l>1 \tag{15}$$

The first of these equations gives us

$$B_0=C_0 \tag{16}$$

Substituting this into 8 gives us $A_0=0$; $B_0=aV_0=C_0$.

The second equation can be reduced by substituting for B_1 and C_1 in terms of A_1 from the above relations:

$$\left(2\frac{b^3-a^3}{b^3}+1+2\frac{a^3}{b^3}\right)A_1 = \frac{k}{\varepsilon_0} \tag{17}$$

$$A_1 = \frac{k}{3\varepsilon_0} \tag{18}$$

This gives us, from the above relations

$$B_1 = -\frac{a^3 k}{3\varepsilon_0} \tag{19}$$

$$C_1 = \frac{k}{3\varepsilon_0}\left(b^3-a^3\right) \tag{20}$$

The third equation can be solved by taking $A_l = B_l = C_l = 0$ for all $l > 1$. Since the solution to Laplace's equation is unique, this must be *the* solution. Putting all this together, we get:

$$V(r,\theta) = \begin{cases} \frac{aV_0}{r} + \frac{k}{3\varepsilon_0}\left(r - \frac{a^3}{r^2}\right)\cos\theta & a \le r \le b \\ \frac{aV_0}{r} + \frac{k}{3\varepsilon_0}\left(\frac{b^3-a^3}{r^2}\right)\cos\theta & r > b \end{cases} \tag{21}$$

The induced surface charge on the conductor can be found from the normal derivative at the surface $r = a$ (the potential inside the conductor is constant, so its derivative is zero):

$$-\frac{\sigma}{\varepsilon_0} = \left.\frac{\partial V}{\partial r}\right|_{r=a} \tag{22}$$

$$= -\frac{V_0}{a} + \frac{k}{\varepsilon_0}\cos\theta \tag{23}$$

$$\sigma = \frac{V_0\varepsilon_0}{a} - k\cos\theta \tag{24}$$

If the sphere is grounded so that $V_0 = 0$, then the induced charge is just the negative of the charge on the shell. The total charge in the system is thus the surface area of the conducting sphere times the constant term:

$$Q = 4\pi a^2 \frac{V_0\varepsilon_0}{a} \tag{25}$$

$$= 4\pi\varepsilon_0 a V_0 \tag{26}$$

For large distances, from the above formula

$$V \to \frac{aV_0}{r} \tag{27}$$

This is consistent with the total charge, since the potential for a point charge is

$$V = \frac{Q}{4\pi\varepsilon_0 r} \tag{28}$$

$$= \frac{aV_0}{r} \tag{29}$$