

## LAPLACE'S EQUATION: CONDUCTING SPHERE AND SHELL OF CHARGE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 3.37.

Another example of using the series solution to Laplace's equation to find the potential using boundary conditions.

We have a conducting sphere of radius  $a$  maintained at a constant potential  $V_0$ . Concentric with this sphere is a larger spherical shell of radius  $b$  with a surface charge density of  $\sigma(\theta) = k \cos \theta$ . Find the potential in the two regions (i)  $a \leq r \leq b$  and (ii)  $r > b$ .

The general solution in terms of Legendre polynomials is

$$(0.1) \quad V(r, \theta) = \sum_{l=0}^{\infty} \left[ A_l r^l + \frac{B_l}{r^{l+1}} \right] P_l(\cos \theta)$$

In the inner region, we must use the most general form of the solution, since  $r$  tends neither to zero nor infinity. In the outer region, to keep the potential finite, we dispose of the terms in  $r^l$ , so we have

$$(0.2) \quad V_{out}(r, \theta) = \sum_{l=0}^{\infty} \frac{C_l}{r^{l+1}} P_l(\cos \theta)$$

Note that we must use a different set of coefficients since the potential will have a different functional form outside the shell.

We can now apply the various boundary conditions. First, at  $r = a$  we must have  $V = V_0$  so

$$(0.3) \quad \sum_{l=0}^{\infty} \left[ A_l a^l + \frac{B_l}{a^{l+1}} \right] P_l(\cos \theta) = V_0$$

From the orthogonality of the  $P_l$ , only the  $l = 0$  term on the left is non-zero, so we get

$$(0.4) \quad A_0 + \frac{B_0}{a} = V_0$$

$$(0.5) \quad B_l = -a^{2l+1} A_l \quad (l > 0)$$

Next, we look at  $r = b$ . Here, the potential must be continuous, so

$$(0.6) \quad \sum_{l=0}^{\infty} \left[ A_l b^l + \frac{B_l}{b^{l+1}} \right] P_l(\cos \theta) = \sum_{l=0}^{\infty} \frac{C_l}{r^{l+1}} P_l(\cos \theta)$$

Equating coefficients of the  $P_l$  we get

$$(0.7) \quad A_l + \frac{B_l}{b^{l+1}} = \frac{C_l}{b^{l+1}}$$

Using the above relation between  $A_l$  and  $B_l$  we get, first for  $l = 0$ :

$$(0.8) \quad A_0 + \frac{B_0}{b} = \frac{C_0}{b}$$

and for  $l > 0$ :

$$(0.9) \quad A_l \left( b^l - \frac{a^{2l+1}}{b^{l+1}} \right) = \frac{C_l}{b^{l+1}}$$

$$(0.10) \quad C_l = A_l \left( b^{2l+1} - a^{2l+1} \right)$$

Finally, we use the surface charge density on the shell. From our earlier example of this type, we get

$$(0.11) \quad \left. \frac{\partial V}{\partial r} \right|_{out} - \left. \frac{\partial V}{\partial r} \right|_{in} = -\frac{\sigma}{\epsilon_0}$$

$$(0.12) \quad \sum_{l=0}^{\infty} \left[ -(l+1) \frac{C_l}{b^{l+2}} - l A_l b^{l-1} + (l+1) \frac{B_l}{b^{l+2}} \right] P_l(\cos \theta) = -\frac{k}{\epsilon_0} \cos \theta$$

Since  $P_1(\cos \theta) = \cos \theta$ , we can equate coefficients on both sides to get

$$(0.13) \quad -\frac{C_0}{b^2} + \frac{B_0}{b^2} = 0$$

$$(0.14) \quad -2 \frac{C_1}{b^3} - A_1 + 2 \frac{B_1}{b^3} = -\frac{k}{\epsilon_0}$$

$$(0.15) \quad -(l+1) \frac{C_l}{b^{l+2}} - l A_l b^{l-1} + (l+1) \frac{B_l}{b^{l+2}} = 0 \quad l > 1$$

The first of these equations gives us

$$(0.16) \quad B_0 = C_0$$

Substituting this into 0.8 gives us  $A_0 = 0$ ;  $B_0 = aV_0 = C_0$ .

The second equation can be reduced by substituting for  $B_1$  and  $C_1$  in terms of  $A_1$  from the above relations:

$$(0.17) \quad \left(2\frac{b^3 - a^3}{b^3} + 1 + 2\frac{a^3}{b^3}\right)A_1 = \frac{k}{\epsilon_0}$$

$$(0.18) \quad A_1 = \frac{k}{3\epsilon_0}$$

This gives us, from the above relations

$$(0.19) \quad B_1 = -\frac{a^3 k}{3\epsilon_0}$$

$$(0.20) \quad C_1 = \frac{k}{3\epsilon_0}(b^3 - a^3)$$

The third equation can be solved by taking  $A_l = B_l = C_l = 0$  for all  $l > 1$ . Since the solution to Laplace's equation is unique, this must be *the* solution. Putting all this together, we get:

$$(0.21) \quad V(r, \theta) = \begin{cases} \frac{aV_0}{r} + \frac{k}{3\epsilon_0} \left(r - \frac{a^3}{r^2}\right) \cos \theta & a \leq r \leq b \\ \frac{aV_0}{r} + \frac{k}{3\epsilon_0} \left(\frac{b^3 - a^3}{r^2}\right) \cos \theta & r > b \end{cases}$$

The induced surface charge on the conductor can be found from the normal derivative at the surface  $r = a$  (the potential inside the conductor is constant, so its derivative is zero):

$$(0.22) \quad -\frac{\sigma}{\epsilon_0} = \left. \frac{\partial V}{\partial r} \right|_{r=a}$$

$$(0.23) \quad = -\frac{V_0}{a} + \frac{k}{\epsilon_0} \cos \theta$$

$$(0.24) \quad \sigma = \frac{V_0 \epsilon_0}{a} - k \cos \theta$$

If the sphere is grounded so that  $V_0 = 0$ , then the induced charge is just the negative of the charge on the shell. The total charge in the system is thus the surface area of the conducting sphere times the constant term:

$$(0.25) \quad Q = 4\pi a^2 \frac{V_0 \epsilon_0}{a}$$

$$(0.26) \quad = 4\pi \epsilon_0 a V_0$$

For large distances, from the above formula

$$(0.27) \quad V \rightarrow \frac{aV_0}{r}$$

This is consistent with the total charge, since the potential for a point charge is

$$(0.28) \quad V = \frac{Q}{4\pi\epsilon_0 r}$$

$$(0.29) \quad = \frac{aV_0}{r}$$