LAPLACE'S EQUATION - CHARGED LINE SEGMENT

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 3.38.

An example of using the general series solution to Laplace's equation. We have a linear charge density λ along the z axis between $z=-a$ and $z=+a$. Find the potential for any position where $r>a$.

We can approach this problem using the same technique as we used for finding the potential due to a charged disk. We worked out the potential for points in the xy plane earlier (see Example 2 in this earlier post). Using the coordinates suitable for the current problem, we have

$$V(r,\pi/2) = \frac{2\lambda}{4\pi\epsilon_0}\left[\ln\left(a+\sqrt{r^2+a^2}\right)-\ln r\right] \tag{1}$$

$$= \frac{\lambda}{2\pi\epsilon_0}\ln\left[\frac{a}{r}+\sqrt{1+\frac{a^2}{r^2}}\right] \tag{2}$$

We can expand the log term in powers of $1/r$ and get

$$V(r,\pi/2) = \frac{\lambda}{2\pi\epsilon_0}\left[\frac{a}{r}-\frac{1}{6}\frac{a^3}{r^3}+\frac{3}{40}\frac{a^5}{r^5}+\ldots\right] \tag{3}$$

This series must match the general form of Laplace's equation for $r>a$. This consists entirely of the terms in inverse powers of r, since we require the potential to be finite for large r. That is, we must have

$$V(r,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta) \tag{4}$$

By comparing the series, we see that terms in odd l must vanish, so $B_l=0$ if l is odd. The second series must match the earlier one for the special case of $\theta=\pi/2$, that is when $\cos\theta=0$. Looking up tables of the Legendre polynomials, we find for the first few:

$$P_0(0) = 1 \tag{5}$$
$$P_2(0) = -\frac{1}{2} \tag{6}$$
$$P_4(0) = \frac{3}{8} \tag{7}$$

Therefore, we have

$$V(r,\pi/2) = \frac{B_0}{r} - \frac{B_2}{2r^3} + \frac{3}{8}\frac{B_4}{r^5} + \ldots \tag{8}$$

Comparing the two series, we get

$$B_0 = \frac{\lambda a}{2\pi\epsilon_0} \tag{9}$$
$$B_2 = \frac{\lambda}{2\pi\epsilon_0}\frac{a^3}{3} \tag{10}$$
$$B_4 = \frac{\lambda}{2\pi\epsilon_0}\frac{a^5}{5} \tag{11}$$

The general solution is then

$$V(r,\theta) = \frac{\lambda}{2\pi\epsilon_0}\left[\frac{a}{r} + \frac{a^3}{3r^3}P_2(\cos\theta) + \frac{a^5}{5r^5}P_4(\cos\theta) + \ldots\right] \tag{12}$$
$$= \frac{\lambda}{2\pi\epsilon_0}\frac{a}{r}\left[1 + \frac{a^2}{3r^2}P_2(\cos\theta) + \frac{a^4}{5r^4}P_4(\cos\theta) + \ldots\right] \tag{13}$$

We can write this in terms of the total charge in the line segment, which is $Q = 2a\lambda$:

$$V(r,\theta) = \frac{Q}{4\pi\epsilon_0 r}\left[1 + \frac{a^2}{3r^2}P_2(\cos\theta) + \frac{a^4}{5r^4}P_4(\cos\theta) + \ldots\right] \tag{14}$$