

LAPLACE'S EQUATION - CHARGED LINE SEGMENT

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 3.38.

An example of using the general series solution to Laplace's equation. We have a linear charge density λ along the z axis between $z = -a$ and $z = +a$. Find the potential for any position where $r > a$.

We can approach this problem using the same technique as we used for finding the potential due to a charged disk. We worked out the potential for points in the xy plane earlier (see Example 2 in this earlier post). Using the coordinates suitable for the current problem, we have

$$\begin{aligned} (1) \quad V(r, \pi/2) &= \frac{2\lambda}{4\pi\epsilon_0} \left[\ln \left(a + \sqrt{r^2 + a^2} \right) - \ln r \right] \\ (2) \quad &= \frac{\lambda}{2\pi\epsilon_0} \ln \left[\frac{a}{r} + \sqrt{1 + \frac{a^2}{r^2}} \right] \end{aligned}$$

We can expand the log term in powers of $1/r$ and get

$$(3) \quad V(r, \pi/2) = \frac{\lambda}{2\pi\epsilon_0} \left[\frac{a}{r} - \frac{1}{6} \frac{a^3}{r^3} + \frac{3}{40} \frac{a^5}{r^5} + \dots \right]$$

This series must match the general form of Laplace's equation for $r > a$. This consists entirely of the terms in inverse powers of r , since we require the potential to be finite for large r . That is, we must have

$$(4) \quad V(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

By comparing the series, we see that terms in odd l must vanish, so $B_l = 0$ if l is odd. The second series must match the earlier one for the special case of $\theta = \pi/2$, that is when $\cos \theta = 0$. Looking up tables of the Legendre polynomials, we find for the first few:

$$(5) \quad P_0(0) = 1$$

$$(6) \quad P_2(0) = -\frac{1}{2}$$

$$(7) \quad P_4(0) = \frac{3}{8}$$

Therefore, we have

$$(8) \quad V(r, \pi/2) = \frac{B_0}{r} - \frac{B_2}{2r^3} + \frac{3B_4}{8r^5} + \dots$$

Comparing the two series, we get

$$(9) \quad B_0 = \frac{\lambda a}{2\pi\epsilon_0}$$

$$(10) \quad B_2 = \frac{\lambda a^3}{2\pi\epsilon_0 3}$$

$$(11) \quad B_4 = \frac{\lambda a^5}{2\pi\epsilon_0 5}$$

The general solution is then

$$(12) \quad V(r, \theta) = \frac{\lambda}{2\pi\epsilon_0} \left[\frac{a}{r} + \frac{a^3}{3r^3} P_2(\cos \theta) + \frac{a^5}{5r^5} P_4(\cos \theta) + \dots \right]$$

$$(13) \quad = \frac{\lambda}{2\pi\epsilon_0} \frac{a}{r} \left[1 + \frac{a^2}{3r^2} P_2(\cos \theta) + \frac{a^4}{5r^4} P_4(\cos \theta) + \dots \right]$$

We can write this in terms of the total charge in the line segment, which is $Q = 2a\lambda$:

$$(14) \quad V(r, \theta) = \frac{Q}{4\pi\epsilon_0 r} \left[1 + \frac{a^2}{3r^2} P_2(\cos \theta) + \frac{a^4}{5r^4} P_4(\cos \theta) + \dots \right]$$