LAPLACE'S EQUATION - CYLINDRICAL SHELL WITH OPPOSING CHARGES

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 3.39.

As another example of applying the solution to Laplace's equation in cylindrical coordinates, we consider the following problem. We are given a cylindrical non-conducting shell or radius R carrying a charge density of σ_0 on its upper half $(-\pi < \phi < 0)$ and $-\sigma_0$ on its lower half $(0 < \phi < \pi)$. Find the potential everywhere.

We can begin in the same manner as for the other problem involving a cylindrical shell that we solved earlier. The solution is the same up until the point where we introduce the surface charge.

Thus, outside the shell, we have

$$V_{out} = B_{out} + \sum_{n=1}^{\infty} \left[\frac{D_n}{r^n} \cos n\phi - \frac{C_n}{r^n} \sin n\phi \right] \tag{1}$$

Inside the shell, we have

$$V_{in} = B_{in} + \sum_{n=1}^{\infty} \left[A_n r^n \sin n\phi + B_n r^n \cos n\phi \right] \tag{2}$$

Since the potential is continuous over a surface charge, we must have $V_{out}(R) = V_{in}(R)$, so we get

$$B_{out} + \sum_{n=1}^{\infty} \left[\frac{D_n}{R^n} \cos n\phi - \frac{C_n}{R^n} \sin n\phi \right] = B_{in} + \sum_{n=1}^{\infty} \left[A_n R^n \sin n\phi + B_n R^n \cos n\phi \right] \tag{3}$$

Equating coefficients of the sine and cosine, we get

$$B_{out} = B_{in} \tag{4}$$

$$C_n = -A_n R^{2n} \tag{5}$$

$$D_n = B_n R^{2n} \tag{6}$$

The outward derivative of the potential is discontinuous across a surface charge, and we have

$$\left.\frac{\partial V}{\partial r}\right|_{out}-\left.\frac{\partial V}{\partial r}\right|_{in}=-\frac{\sigma}{\varepsilon_0} \tag{7}$$

Plugging in the formulas for V_{out} and V_{in}, we get

$$\sum_{n=1}^{\infty}\left[\frac{-nR^{2n}A_n}{R^{n+1}}-nR^{n-1}A_n\right]\sin n\phi+\sum_{n=1}^{\infty}\left[\frac{-nR^{2n}B_n}{R^{n+1}}-nR^{n-1}B_n\right]\cos n\phi=\begin{cases}\frac{\sigma_0}{\varepsilon_0} & -\pi<\phi<0\\ -\frac{\sigma_0}{\varepsilon_0} & 0<\phi<\pi\end{cases} \tag{8}$$

Since the surface charge is an odd function of ϕ, we can eliminate the cosine terms, since the cosine is an even function. Therefore, we have $B_n = D_n = 0$. We are free to choose the potential at infinity to be any constant, so we might as well take it to be zero, in which case we have $B_{in} = B_{out} = 0$. We are therefore left with, after simplifying the term in brackets:

$$-2\sum_{n=1}^{\infty} nR^{n-1}A_n\sin n\phi=\begin{cases}\frac{\sigma_0}{\varepsilon_0} & -\pi<\phi<0\\ -\frac{\sigma_0}{\varepsilon_0} & 0<\phi<\pi\end{cases} \tag{9}$$

To find the A_n, we can use the fact that the set of $\sin n\phi$ functions is orthogonal over the interval $[-\pi,\pi]$. That is

$$\int_{-\pi}^{\pi}\sin m\phi\sin n\phi\, d\phi=\begin{cases}0 & n\neq m\\ \pi & n=m\end{cases} \tag{10}$$

Therefore we can multiply both sides by $\sin m\phi$ and integrate to get

$$-2\pi mR^{m-1}A_m = \frac{\sigma_0}{\varepsilon_0}\left[\int_{-\pi}^{0}\sin m\phi\, d\phi-\int_{0}^{\pi}\sin m\phi\, d\phi\right] \tag{11}$$

The integrals in brackets on the right come out to

$$\left[\int_{-\pi}^{0}\sin m\phi\, d\phi-\int_{0}^{\pi}\sin m\phi\, d\phi\right]=\begin{cases}0 & m\text{ even}\\ -\frac{4}{m} & m\text{ odd}\end{cases} \tag{12}$$

Therefore (changing the index from m to n for convenience):

$$A_n=\begin{cases}0 & n\text{ even}\\ \frac{2\sigma_0}{\pi\varepsilon_0 n^2R^{n-1}} & n\text{ odd}\end{cases} \tag{13}$$

We thus get

$$C_n=\begin{cases}0 & n\text{ even}\\ -\frac{2\sigma_0 R^{n+1}}{\pi\varepsilon_0 n^2} & n\text{ odd}\end{cases} \tag{14}$$

The final formula for the potential is

$$V(r,\phi)=\begin{cases}\frac{2\sigma_0}{\pi\epsilon_0}\sum_{n\,odd}^{\infty}\frac{r^n}{n^2R^{n-1}}\sin n\phi & r<R\\ \frac{2\sigma_0}{\pi\epsilon_0}\sum_{n\,odd}^{\infty}\frac{R^{n+1}}{n^2r^n}\sin n\phi & r>R\end{cases} \tag{15}$$