LAPLACE'S EQUATION - CYLINDRICAL SHELL WITH OPPOSING CHARGES

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 3.39.

As another example of applying the solution to Laplace's equation in cylindrical coordinates, we consider the following problem. We are given a cylindrical non-conducting shell or radius R carrying a charge density of σ_0 on its upper half $(-\pi<\phi<0)$ and $-\sigma_0$ on its lower half $(0<\phi<\pi)$. Find the potential everywhere.

We can begin in the same manner as for the other problem involving a cylindrical shell that we solved earlier. The solution is the same up until the point where we introduce the surface charge.

Thus, outside the shell, we have

$$V_{out}=B_{out}+\sum_{n=1}^{\infty}\left[\frac{D_n}{r^n}\cos n\phi-\frac{C_n}{r^n}\sin n\phi\right] \tag{1}$$

Inside the shell, we have

$$V_{in}=B_{in}+\sum_{n=1}^{\infty}\left[A_n r^n\sin n\phi+B_n r^n\cos n\phi\right] \tag{2}$$

Since the potential is continuous over a surface charge, we must have $V_{out}(R)=V_{in}(R)$, so we get

$$B_{out}+\sum_{n=1}^{\infty}\left[\frac{D_n}{R^n}\cos n\phi-\frac{C_n}{R^n}\sin n\phi\right]=B_{in}+\sum_{n=1}^{\infty}\left[A_n R^n\sin n\phi+B_n R^n\cos n\phi\right] \tag{3}$$

Equating coefficients of the sine and cosine, we get

$$\begin{aligned} B_{out} &= B_{in} && (4)\\ C_n &= -A_n R^{2n} && (5)\\ D_n &= B_n R^{2n} && (6)\end{aligned}$$

The outward derivative of the potential is discontinuous across a surface charge, and we have

$$\left.\frac{\partial V}{\partial r}\right|_{out} - \left.\frac{\partial V}{\partial r}\right|_{in} = -\frac{\sigma}{\varepsilon_0} \tag{7}$$

Plugging in the formulas for V_{out} and V_{in}, we get

$$\sum_{n=1}^{\infty}\left[\frac{-nR^{2n}A_n}{R^{n+1}} - nR^{n-1}A_n\right]\sin n\phi + \sum_{n=1}^{\infty}\left[\frac{-nR^{2n}B_n}{R^{n+1}} - nR^{n-1}B_n\right]\cos n\phi = \begin{cases}\frac{\sigma_0}{\varepsilon_0} & -\pi < \phi < 0\\ -\frac{\sigma_0}{\varepsilon_0} & 0 < \phi < \pi\end{cases} \tag{8}$$

Since the surface charge is an odd function of ϕ, we can eliminate the cosine terms, since the cosine is an even function. Therefore, we have $B_n = D_n = 0$. We are free to choose the potential at infinity to be any constant, so we might as well take it to be zero, in which case we have $B_{in} = B_{out} = 0$. We are therefore left with, after simplifying the term in brackets:

$$-2\sum_{n=1}^{\infty} nR^{n-1}A_n \sin n\phi = \begin{cases}\frac{\sigma_0}{\varepsilon_0} & -\pi < \phi < 0\\ -\frac{\sigma_0}{\varepsilon_0} & 0 < \phi < \pi\end{cases} \tag{9}$$

To find the A_n, we can use the fact that the set of $\sin n\phi$ functions is orthogonal over the interval $[-\pi,\pi]$. That is

$$\int_{-\pi}^{\pi} \sin m\phi \sin n\phi \, d\phi = \begin{cases}0 & n \neq m\\ \pi & n = m\end{cases} \tag{10}$$

Therefore we can multiply both sides by $\sin m\phi$ and integrate to get

$$-2\pi m R^{m-1}A_m = \frac{\sigma_0}{\varepsilon_0}\left[\int_{-\pi}^{0} \sin m\phi \, d\phi - \int_{0}^{\pi} \sin m\phi \, d\phi\right] \tag{11}$$

The integrals in brackets on the right come out to

$$\left[\int_{-\pi}^{0} \sin m\phi \, d\phi - \int_{0}^{\pi} \sin m\phi \, d\phi\right] = \begin{cases}0 & m \text{ even}\\ -\frac{4}{m} & m \text{ odd}\end{cases} \tag{12}$$

Therefore (changing the index from m to n for convenience):

$$A_n = \begin{cases}0 & n \text{ even}\\ \frac{2\sigma_0}{\pi\varepsilon_0 n^2 R^{n-1}} & n \text{ odd}\end{cases} \tag{13}$$

We thus get

$$C_n = \begin{cases} 0 & n \text{ even} \\ -\frac{2\sigma_0 R^{n+1}}{\pi\varepsilon_0 n^2} & n \text{ odd} \end{cases} \tag{14}$$

The final formula for the potential is

$$V(r,\phi) = \begin{cases} \frac{2\sigma_0}{\pi\varepsilon_0} \sum_{n\ odd}^{\infty} \frac{r^n}{n^2 R^{n-1}} \sin n\phi & r < R \\ \frac{2\sigma_0}{\pi\varepsilon_0} \sum_{n\ odd}^{\infty} \frac{R^{n+1}}{n^2 r^n} \sin n\phi & r > R \end{cases} \tag{15}$$