

## LAPLACE'S EQUATION - CYLINDRICAL SHELL WITH OPPOSING CHARGES

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 3.39.

As another example of applying the solution to Laplace's equation in cylindrical coordinates, we consider the following problem. We are given a cylindrical non-conducting shell of radius  $R$  carrying a charge density of  $\sigma_0$  on its upper half ( $-\pi < \phi < 0$ ) and  $-\sigma_0$  on its lower half ( $0 < \phi < \pi$ ). Find the potential everywhere.

We can begin in the same manner as for the other problem involving a cylindrical shell that we solved earlier. The solution is the same up until the point where we introduce the surface charge.

Thus, outside the shell, we have

$$(0.1) \quad V_{out} = B_{out} + \sum_{n=1}^{\infty} \left[ \frac{D_n}{r^n} \cos n\phi - \frac{C_n}{r^n} \sin n\phi \right]$$

Inside the shell, we have

$$(0.2) \quad V_{in} = B_{in} + \sum_{n=1}^{\infty} [A_n r^n \sin n\phi + B_n r^n \cos n\phi]$$

Since the potential is continuous over a surface charge, we must have  $V_{out}(R) = V_{in}(R)$ , so we get

$$(0.3) \quad B_{out} + \sum_{n=1}^{\infty} \left[ \frac{D_n}{R^n} \cos n\phi - \frac{C_n}{R^n} \sin n\phi \right] = B_{in} + \sum_{n=1}^{\infty} [A_n R^n \sin n\phi + B_n R^n \cos n\phi]$$

Equating coefficients of the sine and cosine, we get

$$(0.4) \quad B_{out} = B_{in}$$

$$(0.5) \quad C_n = -A_n R^{2n}$$

$$(0.6) \quad D_n = B_n R^{2n}$$

The outward derivative of the potential is discontinuous across a surface charge, and we have

$$(0.7) \quad \left. \frac{\partial V}{\partial r} \right|_{out} - \left. \frac{\partial V}{\partial r} \right|_{in} = -\frac{\sigma}{\epsilon_0}$$

Plugging in the formulas for  $V_{out}$  and  $V_{in}$ , we get

$$(0.8) \quad \sum_{n=1}^{\infty} \left[ \frac{-nR^{2n}A_n}{R^{n+1}} - nR^{n-1}A_n \right] \sin n\phi + \sum_{n=1}^{\infty} \left[ \frac{-nR^{2n}B_n}{R^{n+1}} - nR^{n-1}B_n \right] \cos n\phi = \begin{cases} \frac{\sigma_0}{\epsilon_0} & -\pi < \phi < 0 \\ -\frac{\sigma_0}{\epsilon_0} & 0 < \phi < \pi \end{cases}$$

Since the surface charge is an odd function of  $\phi$ , we can eliminate the cosine terms, since the cosine is an even function. Therefore, we have  $B_n = D_n = 0$ . We are free to choose the potential at infinity to be any constant, so we might as well take it to be zero, in which case we have  $B_{in} = B_{out} = 0$ . We are therefore left with, after simplifying the term in brackets:

$$(0.9) \quad -2 \sum_{n=1}^{\infty} nR^{n-1}A_n \sin n\phi = \begin{cases} \frac{\sigma_0}{\epsilon_0} & -\pi < \phi < 0 \\ -\frac{\sigma_0}{\epsilon_0} & 0 < \phi < \pi \end{cases}$$

To find the  $A_n$ , we can use the fact that the set of  $\sin n\phi$  functions is orthogonal over the interval  $[-\pi, \pi]$ . That is

$$(0.10) \quad \int_{-\pi}^{\pi} \sin m\phi \sin n\phi d\phi = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$$

Therefore we can multiply both sides by  $\sin m\phi$  and integrate to get

$$(0.11) \quad -2\pi mR^{m-1}A_m = \frac{\sigma_0}{\epsilon_0} \left[ \int_{-\pi}^0 \sin m\phi d\phi - \int_0^{\pi} \sin m\phi d\phi \right]$$

The integrals in brackets on the right come out to

$$(0.12) \quad \left[ \int_{-\pi}^0 \sin m\phi d\phi - \int_0^{\pi} \sin m\phi d\phi \right] = \begin{cases} 0 & m \text{ even} \\ -\frac{4}{m} & m \text{ odd} \end{cases}$$

Therefore (changing the index from  $m$  to  $n$  for convenience):

$$(0.13) \quad A_n = \begin{cases} 0 & n \text{ even} \\ \frac{2\sigma_0}{\pi\epsilon_0 n^2 R^{n-1}} & n \text{ odd} \end{cases}$$

We thus get

$$(0.14) \quad C_n = \begin{cases} 0 & n \text{ even} \\ -\frac{2\sigma_0 R^{n+1}}{\pi\epsilon_0 n^2} & n \text{ odd} \end{cases}$$

The final formula for the potential is

$$(0.15) \quad V(r, \phi) = \begin{cases} \frac{2\sigma_0}{\pi\epsilon_0} \sum_{n \text{ odd}}^{\infty} \frac{r^n}{n^2 R^{n-1}} \sin n\phi & r < R \\ \frac{2\sigma_0}{\pi\epsilon_0} \sum_{n \text{ odd}}^{\infty} \frac{R^{n+1}}{n^2 r^n} \sin n\phi & r > R \end{cases}$$