

MULTIPOLE EXPANSION OF LINEAR CHARGE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 3.40.

This is a rather specialized example of the multipole expansion. We are given a linear charge distribution on the z axis between $z = -a$ and $z = +a$. For each of three such distributions, find the leading term in the multipole expansion.

We saw in the post where we derived the multipole expansion that

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int \rho(\mathbf{r}') P_n(\cos\theta') r'^n d^3\mathbf{r}' \quad (1)$$

where ρ is the volume charge density and P_n is a Legendre polynomial.

When using this formula, it is crucial to understand what the vectors \mathbf{r} and \mathbf{r}' , and the angle θ' refer to. The vector \mathbf{r} is the vector from the origin to the point at which we wish to calculate the potential. It can be in the region where the charge density ρ is non-zero or outside it. The integration variable \mathbf{r}' is the vector from the origin to the volume element at which we measure the charge density $\rho(\mathbf{r}')$. In practice, \mathbf{r}' extends only over that region of space where there is charge, that is, where $\rho(\mathbf{r}') \neq 0$. Finally, the angle θ' is the angle between \mathbf{r} and \mathbf{r}' . Inside the integral, θ' varies as \mathbf{r}' varies over the region where there is charge, and in practice we would need to work out the relation between θ' and \mathbf{r}' in order to do the integral.

For the special case in this example, however, we can simplify things a bit by considering the geometry of the situation. Since the charge lies entirely on the z axis between $z = -a$ and $z = +a$, then the angle θ' is the same for all charge lying in the interval $0 < z \leq +a$. For charge in the interval $-a \leq z < 0$, the angle between \mathbf{r} and \mathbf{r}' is $\pi - \theta'$. Since $\cos(\pi - \theta') = -\cos\theta'$ we have for the Legendre polynomials in the region $z < 0$

$$P_n(\cos(\pi - \theta')) = P_n(-\cos\theta') \quad (2)$$

$$= (-1)^n P_n(\cos\theta') \quad (3)$$

since P_n is even (odd) if n is even (odd).

Now if we remember that \mathbf{r}' is the vector from the origin to a charge element, then in the case where all the charge resides on the z axis, we have $r' = |z|$. For $z < 0$, we therefore have $(r')^n = (-1)^n z^n$. If we combine these two results, then we see that for $z < 0$

$$P_n(\cos(\pi - \theta'))r'^n = (-1)^n P_n(\cos \theta')(-1)^n z^n \quad (4)$$

$$= P_n(\cos \theta')z^n \quad (5)$$

That is, due to the two negatives cancelling out, the product of these two terms is the same over the entire interval $-a < z < a$.

For a given value of \mathbf{r} (that is, for a given observation point), θ' is constant over the entire charge distribution, so we can take $P_n(\cos \theta')$ outside the integral, and we end up with

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{P_n(\cos \theta)}{r^{n+1}} \int_{-a}^a z^n \lambda(z) dz \quad (6)$$

where we've used the more traditional symbol λ to denote a linear charge density, and we've dropped the prime off θ since this variable depends only on \mathbf{r} and not on the charge location.

Note that this formula works *only* for the special case where we have a linear charge distributed along the z axis. That's why we said that this a very specialized situation.

Now we need only specify $\rho(z)$ and work out the integrals. We'll consider three cases, and find only the first non-zero term in each case.

(a) $\lambda = k \cos(\pi z/2a)$. In this case the leading term is $n = 0$, since

$$k \int_{-a}^a \cos(\pi z/2a) dz = \frac{4ka}{\pi} \quad (7)$$

so

$$V_0 = \frac{1}{4\pi\epsilon_0} \frac{P_0(\cos \theta)}{r} \frac{4ka}{\pi} \quad (8)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{4ka}{\pi r} \quad (9)$$

In this case, the monopole term dominates.

(b) $\lambda = k \sin(\pi z/a)$. In this case the total charge is zero, so we look at the dipole ($n = 1$) term.

$$\int_{-a}^a z\lambda(z)dz = k \int_{-a}^a z \sin(\pi z/a)dz \quad (10)$$

$$= \frac{2ka^2}{\pi} \quad (11)$$

so

$$V_1 = \frac{1}{4\pi\epsilon_0} \frac{P_1(\cos\theta)}{r^2} \frac{2ka^2}{\pi} \quad (12)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{2ka^2}{\pi r^2} \cos\theta \quad (13)$$

(c) $\lambda = k \cos(\pi z/a)$. In this case, the monopole and dipole terms are both zero, so we look at the quadrupole term.

$$\int_{-a}^a z^2\lambda(z)dz = k \int_{-a}^a z^2 \cos(\pi z/a)dz \quad (14)$$

$$= -\frac{4ka^3}{\pi^2} \quad (15)$$

so

$$V_2 = -\frac{1}{4\pi\epsilon_0} \frac{P_2(\cos\theta)}{r^3} \frac{4ka^3}{\pi^2} \quad (16)$$

$$= -\frac{1}{4\pi\epsilon_0} \frac{2ka^3}{\pi^2 r^3} (3\cos^2\theta - 1) \quad (17)$$