

## MULTIPOLE EXPANSION OF LINEAR CHARGE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 3.40.

This is a rather specialized example of the multipole expansion. We are given a linear charge distribution on the  $z$  axis between  $z = -a$  and  $z = +a$ . For each of three such distributions, find the leading term in the multipole expansion.

We saw in the post where we derived the multipole expansion that

$$(1) \quad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int \rho(\mathbf{r}') P_n(\cos \theta') r'^n d^3\mathbf{r}'$$

where  $\rho$  is the volume charge density and  $P_n$  is a Legendre polynomial.

When using this formula, it is crucial to understand what the vectors  $\mathbf{r}$  and  $\mathbf{r}'$ , and the angle  $\theta'$  refer to. The vector  $\mathbf{r}$  is the vector from the origin to the point at which we wish to calculate the potential. It can be in the region where the charge density  $\rho$  is non-zero or outside it. The integration variable  $\mathbf{r}'$  is the vector from the origin to the volume element at which we measure the charge density  $\rho(\mathbf{r}')$ . In practice,  $\mathbf{r}'$  extends only over that region of space where there is charge, that is, where  $\rho(\mathbf{r}') \neq 0$ . Finally, the angle  $\theta'$  is the angle between  $\mathbf{r}$  and  $\mathbf{r}'$ . Inside the integral,  $\theta'$  varies as  $\mathbf{r}'$  varies over the region where there is charge, and in practice we would need to work out the relation between  $\theta'$  and  $\mathbf{r}'$  in order to do the integral.

For the special case in this example, however, we can simplify things a bit by considering the geometry of the situation. Since the charge lies entirely on the  $z$  axis between  $z = -a$  and  $z = +a$ , then the angle  $\theta'$  is the same for all charge lying in the interval  $0 < z \leq +a$ . For charge in the interval  $-a \leq z < 0$ , the angle between  $\mathbf{r}$  and  $\mathbf{r}'$  is  $\pi - \theta'$ . Since  $\cos(\pi - \theta') = -\cos \theta'$  we have for the Legendre polynomials in the region  $z < 0$

$$(2) \quad P_n(\cos(\pi - \theta')) = P_n(-\cos \theta')$$

$$(3) \quad = (-1)^n P_n(\cos \theta')$$

since  $P_n$  is even (odd) if  $n$  is even (odd).

Now if we remember that  $\mathbf{r}'$  is the vector from the origin to a charge element, then in the case where all the charge resides on the  $z$  axis, we have  $r' = |z|$ . For  $z < 0$ , we therefore have  $(r')^n = (-1)^n z^n$ . If we combine these two results, then we see that for  $z < 0$

$$(4) \quad P_n(\cos(\pi - \theta')) r'^n = (-1)^n P_n(\cos \theta') (-1)^n z^n$$

$$(5) \quad = P_n(\cos \theta') z^n$$

That is, due to the two negatives cancelling out, the product of these two terms is the same over the entire interval  $-a < z < a$ .

For a given value of  $\mathbf{r}$  (that is, for a given observation point),  $\theta'$  is constant over the entire charge distribution, so we can take  $P_n(\cos \theta')$  outside the integral, and we end up with

$$(6) \quad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{P_n(\cos \theta)}{r^{n+1}} \int_{-a}^a z^n \lambda(z) dz$$

where we've used the more traditional symbol  $\lambda$  to denote a linear charge density, and we've dropped the prime off  $\theta$  since this variable depends only on  $\mathbf{r}$  and not on the charge location.

Note that this formula works *only* for the special case where we have a linear charge distributed along the  $z$  axis. That's why we said that this a very specialized situation.

Now we need only specify  $\rho(z)$  and work out the integrals. We'll consider three cases, and find only the first non-zero term in each case.

(a)  $\lambda = k \cos(\pi z/2a)$ . In this case the leading term is  $n = 0$ , since

$$(7) \quad k \int_{-a}^a \cos(\pi z/2a) dz = \frac{4ka}{\pi}$$

so

$$(8) \quad V_0 = \frac{1}{4\pi\epsilon_0} \frac{P_0(\cos \theta)}{r} \frac{4ka}{\pi}$$

$$(9) \quad = \frac{1}{4\pi\epsilon_0} \frac{4ka}{\pi r}$$

In this case, the monopole term dominates.

(b)  $\lambda = k \sin(\pi z/a)$ . In this case the total charge is zero, so we look at the dipole ( $n = 1$ ) term.

$$(10) \quad \int_{-a}^a z\lambda(z)dz = k \int_{-a}^a z \sin(\pi z/a) dz$$

$$(11) \quad = \frac{2ka^2}{\pi}$$

so

$$(12) \quad V_1 = \frac{1}{4\pi\epsilon_0} \frac{P_1(\cos\theta)}{r^2} \frac{2ka^2}{\pi}$$

$$(13) \quad = \frac{1}{4\pi\epsilon_0} \frac{2ka^2}{\pi r^2} \cos\theta$$

(c)  $\lambda = k \cos(\pi z/a)$ . In this case, the monopole and dipole terms are both zero, so we look at the quadrupole term.

$$(14) \quad \int_{-a}^a z^2\lambda(z)dz = k \int_{-a}^a z^2 \cos(\pi z/a) dz$$

$$(15) \quad = -\frac{4ka^3}{\pi^2}$$

so

$$(16) \quad V_2 = -\frac{1}{4\pi\epsilon_0} \frac{P_2(\cos\theta)}{r^3} \frac{4ka^3}{\pi^2}$$

$$(17) \quad = -\frac{1}{4\pi\epsilon_0} \frac{2ka^3}{\pi^2 r^3} (3\cos^2\theta - 1)$$