MULTIPOLE EXPANSION OF LINEAR CHARGE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 3.40.

This is a rather specialized example of the multipole expansion. We are given a linear charge distribution on the z axis between $z=-a$ and $z=+a$. For each of three such distributions, find the leading term in the multipole expansion.

We saw in the post where we derived the multipole expansion that

$$V(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int\rho(\mathbf{r}')P_n(\cos\theta')r'^n d^3\mathbf{r}' \tag{0.1}$$

where ρ is the volume charge density and P_n is a Legendre polynomial.

When using this formula, it is crucial to understand what the vectors $\mathbf{r}$ and $\mathbf{r}'$, and the angle θ' refer to. The vector $\mathbf{r}$ is the vector from the origin to the point at which we wish to calculate the potential. It can be in the region where the charge density ρ is non-zero or outside it. The integration variable $\mathbf{r}'$ is the vector from the origin to the volume element at which we measure the charge density $\rho(\mathbf{r}')$. In practice, $\mathbf{r}'$ extends only over that region of space where there is charge, that is, where $\rho(\mathbf{r}')\neq 0$. Finally, the angle θ' is the angle between $\mathbf{r}$ and $\mathbf{r}'$. Inside the integral, θ' varies as $\mathbf{r}'$ varies over the region where there is charge, and in practice we would need to work out the relation between θ' and $\mathbf{r}'$ in order to do the integral.

For the special case in this example, however, we can simplify things a bit by considering the geometry of the situation. Since the charge lies entirely on the z axis between $z=-a$ and $z=+a$, then the angle θ' is the same for all charge lying in the interval $0<z\le +a$. For charge in the interval $-a\le z<0$, the angle between $\mathbf{r}$ and $\mathbf{r}'$ is $\pi-\theta'$. Since $\cos(\pi-\theta')=-\cos\theta'$ we have for the Legendre polynomials in the region $z<0$

$$P_n(\cos(\pi-\theta'))=P_n(-\cos\theta') \tag{0.2}$$

$$=(-1)^n P_n(\cos\theta') \tag{0.3}$$

since P_n is even (odd) if n is even (odd).

Now if we remember that $\mathbf{r}'$ is the vector from the origin to a charge element, then in the case where all the charge resides on the z axis, we have $r' = |z|$. For $z < 0$, we therefore have $(r')^n = (-1)^n z^n$. If we combine these two results, then we see that for $z < 0$

$$P_n(\cos(\pi - \theta'))r'^n = (-1)^n P_n(\cos\theta')(-1)^n z^n \tag{0.4}$$

$$= P_n(\cos\theta')z^n \tag{0.5}$$

That is, due to the two negatives cancelling out, the product of these two terms is the same over the entire interval $-a < z < a$.

For a given value of $\mathbf{r}$ (that is, for a given observation point), θ' is constant over the entire charge distribution, so we can take $P_n(\cos\theta')$ outside the integral, and we end up with

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{r^{n+1}}\int_{-a}^{a} z^n \lambda(z)dz \tag{0.6}$$

where we've used the more traditional symbol λ to denote a linear charge density, and we've dropped the prime off θ since this variable depends only on $\mathbf{r}$ and not on the charge location.

Note that this formula works *only* for the special case where we have a linear charge distributed along the z axis. That's why we said that this a very specialized situation.

Now we need only specify $\rho(z)$ and work out the integrals. We'll consider three cases, and find only the first non-zero term in each case.

(a) $\lambda = k\cos(\pi z/2a)$. In this case the leading term is $n = 0$, since

$$k\int_{-a}^{a}\cos(\pi z/2a)dz = \frac{4ka}{\pi} \tag{0.7}$$

so

$$V_0 = \frac{1}{4\pi\varepsilon_0}\frac{P_0(\cos\theta)}{r}\frac{4ka}{\pi} \tag{0.8}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{4ka}{\pi r} \tag{0.9}$$

In this case, the monopole term dominates.

(b) $\lambda = k\sin(\pi z/a)$. In this case the total charge is zero, so we look at the dipole ($n = 1$) term.

$$\int_{-a}^{a} z\lambda(z)dz = k\int_{-a}^{a} z\sin(\pi z/a)dz \tag{0.10}$$

$$= \frac{2ka^2}{\pi} \tag{0.11}$$

so

$$V_1 = \frac{1}{4\pi\varepsilon_0}\frac{P_1(\cos\theta)}{r^2}\frac{2ka^2}{\pi} \tag{0.12}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2ka^2}{\pi r^2}\cos\theta \tag{0.13}$$

(c) $\lambda = k\cos(\pi z/a)$. In this case, the monopole and dipole terms are both zero, so we look at the quadrapole term.

$$\int_{-a}^{a} z^2\lambda(z)dz = k\int_{-a}^{a} z^2\cos(\pi z/a)dz \tag{0.14}$$

$$= -\frac{4ka^3}{\pi^2} \tag{0.15}$$

so

$$V_2 = -\frac{1}{4\pi\varepsilon_0}\frac{P_2(\cos\theta)}{r^3}\frac{4ka^3}{\pi^2} \tag{0.16}$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{2ka^3}{\pi^2 r^3}(3\cos^2\theta - 1) \tag{0.17}$$