

AVERAGE FIELD OVER A SPHERE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 3.41.

The average electric field inside a sphere of radius R due to a point charge q at position \mathbf{r} inside the sphere is the field integrated over the volume of the sphere divided by the sphere's volume:

$$\mathbf{E}_{av} = \frac{1}{4\pi\epsilon_0} \frac{3q}{4\pi R^3} \int \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3} d^3\mathbf{r}' \quad (1)$$

where the integral extends over the interior of the sphere. (Note that the statement of the problem in Griffiths' book has a typo: the unit vector in the integral in part (a) should be script 'r', not \mathbf{r} .)

Now suppose we have a uniformly charged sphere with charge density ρ and wish to find the field at point \mathbf{r} due to this charge. This time the field at \mathbf{r} due to volume element $d^3\mathbf{r}'$ is $(\mathbf{r} - \mathbf{r}') \rho d^3\mathbf{r}' / 4\pi\epsilon_0 |\mathbf{r}' - \mathbf{r}|^3$ so the overall field is

$$\mathbf{E}_\rho = \frac{1}{4\pi\epsilon_0} \int \rho \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3\mathbf{r}' \quad (2)$$

The two fields are the same if we set

$$\rho = -\frac{3q}{4\pi R^3} \quad (3)$$

From Gauss's law, we can work out \mathbf{E}_ρ if we work out the integrals below for a sphere of radius $r < R$:

$$\int \mathbf{E}_\rho \cdot d\mathbf{a} = \frac{1}{\epsilon_0} \int \rho d^3\mathbf{r}' \quad (4)$$

$$4\pi r^2 E_\rho = -\frac{1}{\epsilon_0} \frac{3q}{4\pi R^3} \frac{4\pi r^3}{3} \quad (5)$$

$$E_\rho = -\frac{1}{4\pi\epsilon_0} \frac{qr}{R^3} \quad (6)$$

Since \mathbf{E}_ρ points in the radial direction due to symmetry,

$$\mathbf{E}_\rho = -\frac{1}{4\pi\epsilon_0} \frac{q}{R^3} \mathbf{r} \quad (7)$$

The dipole moment of a single point charge q at position \mathbf{r} is $\mathbf{p} = q\mathbf{r}$, so the field can be written as

$$\mathbf{E}_\rho = \mathbf{E}_{av} = -\frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}}{R^3} \quad (8)$$

From the superposition principle, we can extend this result so that it applies to any distribution of charge within the sphere.

If \mathbf{r} is outside the sphere, the formula for \mathbf{E}_{av} is the same as before, with the integral still extending over the interior of the sphere. The formula for \mathbf{E}_ρ is also the same, but this time if we use Gauss's law and integrate over a spherical surface of radius $r > R$, we get

$$\int \mathbf{E}_\rho \cdot d\mathbf{a} = \frac{1}{\epsilon_0} \int \rho d^3\mathbf{r}' \quad (9)$$

$$4\pi r^2 E_\rho = -\frac{1}{\epsilon_0} \frac{3q}{4\pi R^3} \frac{4\pi R^3}{3} \quad (10)$$

$$E_\rho = -\frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \quad (11)$$

$$\mathbf{E}_\rho = -\frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} \quad (12)$$

The RHS of the second line arises from the fact that all of the charged sphere is now interior to the surface of integration, while on the LHS, we are still integrating over a spherical surface of radius $r > R$. The average field produced by a point charge is thus the same as the field produced by this charge at the centre of the sphere. By superposition we can apply this argument to any distribution of charge external to the sphere.

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