METHOD OF IMAGES - SPHERE IN UNIFORM FIELD

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 3.46.

The method of images can be used to find the potential of the configuration where we have a grounded conducting sphere in a uniform electric field (uniform except around the sphere, of course, where the induced surface charge will distort the field).

We've seen that in the case of a single charge q outside a sphere of radius R, we have

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}} - \frac{1}{\sqrt{(ar/R)^2+R^2-2ra\cos\theta}}\right] \quad (1)$$

where the charge is assumed to be on the z axis at $z = a$ and θ is the usual spherical angle. This solution was found by introducing an image charge $q' = -Rq/a$ at location $z = R^2/a$, and we showed in the earlier post that these two charges produce $V = 0$ at $r = R$, so the boundary condition is the same as in the case of a grounded conducting sphere.

We can now introduce another point charge $-q$ at location $z = -a$, and its image $q'' = Rq/a$ at location $z = -R^2/a$. Since this is just the mirror image of the above configuration, it too will maintain the potential of the sphere at zero, so the boundary condition remains unchanged. The potential due to these two charges is identical to that of the first two, except that the angle is now $\pi-\theta$. Since $\cos(\pi-\theta) = -\cos\theta$, we get

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}} - \frac{1}{\sqrt{(ar/R)^2+R^2-2ra\cos\theta}}\right] \tag{2}$$

$$+\frac{q}{4\pi\epsilon_0}\left[-\frac{1}{\sqrt{r^2+a^2+2ar\cos\theta}} + \frac{1}{\sqrt{(ar/R)^2+R^2+2ra\cos\theta}}\right] \tag{3}$$

$$= \frac{q}{4\pi\epsilon_0 a}\left[\frac{1}{\sqrt{1+r^2/a^2-2(r/a)\cos\theta}} - \frac{1}{\sqrt{1+r^2/a^2+2(r/a)\cos\theta}}\right] \tag{4}$$

$$+\frac{q}{4\pi\epsilon_0 a}\left[-\frac{R}{r}\frac{1}{\sqrt{1+R^4/(ar)^2-2(R^2/ra)\cos\theta}} + \frac{R}{r}\frac{1}{\sqrt{1+R^4/(ar)^2+2(R^2/ra)\cos\theta}}\right] \tag{5}$$

The idea now is to let $a \to \infty$ so the external charges become very far away from the sphere, which makes the electric field due to them essentially uniform. In this approximation, we can use the expansion of $1/\sqrt{1+x} \approx 1-x/2$ to get

$$V(\mathbf{r}) \approx \frac{q}{4\pi\epsilon_0 a}\left[2\frac{r}{a}\cos\theta - 2\frac{R^3}{r^2 a}\cos\theta\right] \tag{6}$$

$$= \frac{2q}{4\pi\epsilon_0 a^2}\left(r - \frac{R^3}{r^2}\right)\cos\theta \tag{7}$$

For $r \to \infty$, we require that the electric field is uniform, with a value of E_0 say. For large r the second term in the above expression can be ignored, and we get

$$V(\mathbf{r}) \to \frac{2q}{4\pi\epsilon_0 a^2} r\cos\theta \tag{8}$$

$$= \frac{2q}{4\pi\epsilon_0 a^2} z \tag{9}$$

Since the field is in the z direction, we have $E_0 = -\partial V/\partial z = -\frac{2q}{4\pi\epsilon_0 a^2}$. That is, this quantity must remain constant as $a \to \infty$, so q must also become infinitely large. This makes sense, since if the charges are infinitely far away, they must be infinitely large to provide a finite field.