LAPLACE'S EQUATION - FOURIER SERIES EXAMPLE 4

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problems 3.47.

Here is another example of solving Laplace's equation in rectangular coordinates. We have a rectangular pipe extending in the z direction with boundaries at $x=\pm b$, $y=0$ and $y=a$. The potential on each face is held constant and satisfies

$$V=\begin{cases}0 & y=0\\ V_0 & y=a\\ 0 & x=-b\\ 0 & x=b\end{cases} \tag{1}$$

The problem is to find $V(x,y)$ everywhere inside the pipe. We can use the separation of variables technique to arrive at a solution of form:

$$V(x,y)=\left(Ae^{kx}+Be^{-kx}\right)(C\sin ky+D\cos ky) \tag{2}$$

Because the problem is symmetric in x we must have $V(-x,y)=V(x,y)$, which means that $A=B$ and we can write

$$V(x,y)=(C\sin ky+D\cos ky)\cosh kx \tag{3}$$

where we've absorbed the constants A and B into C and D.

The boundary condition at $y=0$ means that $D=0$. At first glance we can't do much with the boundary conditions at the other three faces. Let's leave this solution for the moment and try a different approach.

In the original derivation of the separation of variables solution, we had the equations

$$\frac{1}{X}\frac{d^2X}{dx^2} = C_1 \tag{4}$$
$$\frac{1}{Y}\frac{d^2Y}{dy^2} = C_2 \tag{5}$$
$$\frac{1}{Z}\frac{d^2Z}{dz^2} = C_3 \tag{6}$$

We made the implicit assumption that the constants were non-zero, requiring only that their sum is zero so as to satisfy the full Laplace's equation:

$$C_1 + C_2 + C_3 = 0 \tag{7}$$

In this problem, though, we can use a trick to get the solution. Since the problem doesn't depend on z, we can consider the two equations

$$\frac{d^2X}{dx^2} = 0 \tag{8}$$
$$\frac{d^2Y}{dy^2} = 0 \tag{9}$$

The general solution of this pair of equations is linear in each of x and y separately in order for the second derivatives to be zero. Thus we get

$$X(x) = c_1x + c_2 \tag{10}$$
$$Y(y) = c_3y + c_4 \tag{11}$$

and the overall solution is thus

$$V = (c_1x + c_2)(c_3y + c_4) \tag{12}$$

Applying the boundary conditions on y we get

$$V(x,0) = c_4(c_1x + c_2) \tag{13}$$
$$= 0 \tag{14}$$

This must be true for all x so either $c_4 = 0$ or $c_1 = c_2 = 0$. If we choose the latter case, then $V = 0$ everywhere, which isn't much use, so we'll take $c_4 = 0$.

At the other boundary, we have

$$\begin{aligned} V(x,a) &= c_3 a (c_1 x + c_2) && (15)\\ &= V_0 && (16) \end{aligned}$$

Again, this must be true for all x, so $c_1 = 0$ and $c_2 c_3 = V_0/a$. Thus a solution that satisfies the y boundary conditions is

$$V(x,y) = \frac{V_0}{a} y \tag{17}$$

Clearly this doesn't satisfy the x boundary conditions, but we can now return to our earlier solutions to the separation of variables problem to see how to fix this. If we find a solution that satisfies the new boundary conditions

$$V = \begin{cases} 0 & y = 0 \\ 0 & y = a \\ -\frac{V_0}{a} y & x = -b \\ -\frac{V_0}{a} y & x = b \end{cases} \tag{18}$$

then we can add this solution to the solution we just found. This new solution will satisfy all four boundary conditions, since the sums of the potentials at each of the four boundaries come out to what was originally specified. We can now use the usual technique of building an infinite series of solutions and using the orthogonality of the sine functions to work out the coefficients. That is, returning to our first solution, we have, after satisfying the boundary condition at $y = 0$:

$$V(x,y) = C \sin ky \cosh kx \tag{19}$$

If we now require this solution to be zero at $y = a$, then we must have

$$k = \frac{n\pi}{a} \tag{20}$$

for $n = 1, 2, 3, \ldots$. We can then construct the sum of these terms and combine it with the other solution we found to give

$$V(x,y) = \frac{V_0}{a} y + \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) \cosh\left(\frac{n\pi x}{a}\right) \tag{21}$$

To satisfy the x boundary conditions we must have

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) \cosh\left(\frac{n\pi b}{a}\right) = -\frac{V_0}{a} y \tag{22}$$

for $0<y<a$. We now multiply both sides of this equation by $\sin\frac{m\pi y}{a}$ and integrate from 0 to a, using the orthogonality of the sine, that is

$$\int_0^a \sin\frac{m\pi y}{a}\sin\frac{n\pi y}{a}=\begin{cases}0 & n\neq m\\ \frac{a}{2} & n=m\end{cases} \tag{23}$$

Therefore

$$C_n\frac{a}{2}\cosh\frac{n\pi b}{a}=-\frac{V_0}{a}\int_0^a y\sin\frac{n\pi y}{a}dy \tag{24}$$

$$=-\frac{V_0}{a}\left(-\frac{(-1)^n a^2}{\pi n}\right) \tag{25}$$

$$=\frac{(-1)^n aV_0}{\pi n} \tag{26}$$

$$C_n=\frac{2(-1)^n V_0}{n\pi\cosh(n\pi b/a)} \tag{27}$$

The final form of the potential is

$$V(x,y)=\frac{V_0}{a}y+\frac{2V_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\sin(n\pi y/a)\cosh(n\pi x/a)}{n\cosh(n\pi b/a)} \tag{28}$$

PINGBACKS

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