THOMPSON-LAMPARD THEOREM

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problems 3.48a.

In the previous post, we worked out the potential inside a rectangular pipe with three grounded sides and the fourth side held at a constant potential of V_0. The boundary conditions are

$$V=\begin{cases}0 & y=0\\ V_0 & y=a\\ 0 & x=-b\\ 0 & x=b\end{cases} \tag{0.1}$$

The answer is

$$V(x,y)=\frac{V_0}{a}y+\frac{2V_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\sin(n\pi y/a)\cosh(n\pi x/a)}{n\cosh(n\pi b/a)} \tag{0.2}$$

We can now consider a more specialized version of this problem in which the pipe has a square cross-section, so that $b=a/2$. In this case, we have

$$V(x,y)=\frac{V_0}{a}y+\frac{2V_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\sin(n\pi y/a)\cosh(n\pi x/a)}{n\cosh(n\pi/2)} \tag{0.3}$$

We can find the charge density on the face opposite the V_0 face (that is, at $y=0$) by using the usual normal derivative formula:

$$\sigma = -\epsilon_0\frac{\partial V}{\partial n} \tag{0.4}$$

$$= -\epsilon_0\left.\frac{\partial V}{\partial y}\right|_{y=0} \tag{0.5}$$

$$= -\frac{\epsilon_0 V_0}{a}-\frac{2\epsilon_0 V_0}{a}\sum_{n=1}^{\infty}\frac{(-1)^n\cosh(n\pi x/a)}{\cosh(n\pi/2)} \tag{0.6}$$

We can work out from this the charge per unit length for this face in the pipe. Since the charge density depends only on x we can integrate x from $-a/2$ to $a/2$ to get

$$\lambda = \int_{-a/2}^{a/2} \sigma(x)dx \tag{0.7}$$

$$= -\varepsilon_0 V_0 - \frac{2\varepsilon_0 V_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n 2\sinh(n\pi/2)}{n\cosh(n\pi/2)} \tag{0.8}$$

$$= -\varepsilon_0 V_0 \left[1 + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \tanh\left(\frac{n\pi}{2}\right)\right] \tag{0.9}$$

Although I couldn't find any tricky mathematical way of summing the series into a closed form, the geometry of the problem is an instance of the Thompson-Lampard theorem, which applies to any case of 'cross capacitors'; that is two pairs of opposing plates such as we have in this problem. The theorem states that in such a case

$$e^{-\pi C_1/\varepsilon_0} + e^{-\pi C_2/\varepsilon_0} = 1 \tag{0.10}$$

where C_1 and C_2 are the two capacitances per unit length. In our problem, because of the symmetry of the configuration, $C_1 = C_2 \equiv C$ and the formula can then be solved for C:

$$C = \frac{\varepsilon_0}{\pi} \ln 2 \tag{0.11}$$

Since the capacitance is defined as $C = Q/V$ where Q is the charge per unit length on the plates and V is the potential difference between them, we have

$$Q = \frac{\varepsilon_0 V}{\pi} \ln 2 \tag{0.12}$$

If we consider a unit length of the rectangular pipe and look at the pair of plates whose potential differs by V_0 we get

$$Q = \lambda = \frac{\varepsilon_0 V_0}{\pi} \ln 2 \tag{0.13}$$

This is the charge on the plate with positive potential, so the charge on the opposite plate is the negative of this.

The series above converges quite slowly since for large n $\tanh(n\pi/2) \to 1$ and the alternating harmonic series $\sum(-1)^n/n$ converges very slowly. However, we can convert the formula by expressing tanh in its exponential form:

$$\tanh x = \frac{\sinh x}{\cosh x} \tag{0.14}$$

$$= \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{0.15}$$

$$= \frac{e^x + e^{-x} - 2e^{-x}}{e^x + e^{-x}} \tag{0.16}$$

$$= 1 - 2\frac{e^{-x}}{e^x + e^{-x}} \tag{0.17}$$

$$= 1 - 2\frac{1}{1 + e^{2x}} \tag{0.18}$$

So we get

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \tanh\left(\frac{n\pi}{2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left[1 - 2\frac{1}{1+e^{n\pi}}\right] \tag{0.19}$$

The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\ln 2$ (standard calculus result), so we get

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \tanh\left(\frac{n\pi}{2}\right) = -\ln 2 - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{1}{1+e^{n\pi}}\right) \tag{0.20}$$

The remaining series does converge quite fast since the denominator increases exponentially, so we can add up the first few terms numerically and compare the results. Taking the first 20 terms, we get (using software to do the sum)

$$1 + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \tanh\left(\frac{n\pi}{2}\right) \approx 1 - \frac{4\ln 2}{\pi} - \frac{8}{\pi}\sum_{n=1}^{20} \frac{(-1)^n}{n} \left(\frac{1}{1+e^{n\pi}}\right) \tag{0.21}$$

$$\approx 0.2206356003 \tag{0.22}$$

To the same number of significant figures, we have

$$\frac{\ln 2}{\pi} \approx 0.2206356001 \tag{0.23}$$

[Note that the answer given in problem 3.48 in Griffiths's book is wrong - the denominator should be π, not 2.]