POLARIZABILITY OF HYDROGEN - QUANTUM VERSION

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 4.2.

In the last post, we did a crude estimate of the separation between electron and proton in hydrogen due to polarization in an electric field. We assumed there that the electron cloud was a uniformly charged sphere. A more accurate representation is provided by quantum mechanics, which says that the ground state of the electron has a charge density of

$$\rho(r)=\frac{q}{\pi a^{3}}e^{-2r/a} \tag{1}$$

We can use this to provide another estimate of the atomic polarizability. As before, we assume that an electric field separates the electron cloud and proton by a distance d, and that the cloud retains its spherical shape. Using Gauss's law (it's a calculation similar to Example 2 here) we can work out the electric field of a sphere with this charge density at a distance d from its centre. Using the symmetry we can do the integrals over θ and ϕ to get 4π and we are left with the radial integral:

$$4\pi d^{2}E = \frac{q}{\pi a^{3}\epsilon_{0}}\int_{0}^{d}\int_{0}^{\pi}\int_{0}^{2\pi}r^{2}e^{-2r/a}\sin\theta d\phi d\theta dr \tag{2}$$

$$= \frac{4\pi q}{\pi a^{3}\epsilon_{0}}\int_{0}^{d}r^{2}e^{-2r/a}dr \tag{3}$$

This integral can be done twice by parts, or we can just use software, to get

$$d^{2}E=\frac{q}{\pi a^{3}\epsilon_{0}}\left[\frac{a^{3}}{4}-e^{-2d/a}\left(\frac{a^{3}}{4}+\frac{a^{2}}{2}d+\frac{a}{2}d^{2}\right)\right] \tag{4}$$

If we assume that $d\ll a$ (as was the case when we assumed a uniform charge density in the sphere), we can expand this result in a Taylor series (again, by hand or using software) and keep only the leading non-zero term in d. In fact, we discover that all terms up to and including d^{2} cancel out, so the lowest order term that isn't zero is $d^{3}/3$. We therefore have, in this approximation:

$$d^2E \approx \frac{qd^3}{3\pi a^3\epsilon_0} \quad (5)$$

$$E \approx \frac{qd}{3\pi a^3\epsilon_0} \quad (6)$$

We can write this in the usual way using $p = qd$ as the dipole moment, and we have

$$p \approx 3\pi a^3\epsilon_0 E \quad (7)$$

so in this approximation, the polarizability is

$$\alpha = 3\pi a^3\epsilon_0 \quad (8)$$

With the uniformly charged sphere, the answer was $\alpha = 4\pi a^3\epsilon_0$ so there really isn't much difference. However, since the uniformly charged sphere's estimate was already on the low side, this supposedly more accurate form for the electron density is actually worse.

PINGBACKS

Pingback: Polarizability - linear charge distribution