FIELD OF A POLARIZED CYLINDER

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 4.13.

As another example of the use of the bound charges representation of a polarized object, we can look at a cylinder which contains a uniform polarization $\mathbf{P}$ perpendicular to its axis. In this case, $\nabla\cdot\mathbf{P}=0$ everywhere inside the cylinder. If we take the direction of $\mathbf{P}$ to be $\phi=0$ then $\mathbf{P}\cdot\hat{\mathbf{n}}=P\cos\phi$. We thus must find the potential of an infinite cylinder with a surface charge of

$$\sigma_b = P\cos\phi \tag{0.1}$$

We've worked out the general solution to Laplace's equation in cylindrical coordinates before, so we can use the results from there. The solution inside the cylinder is

$$V_{in} = B_{in} + \sum_{n=1}^{\infty}\left[A_n r^n \sin n\phi + B_n r^n \cos n\phi\right] \tag{0.2}$$

while outside it is

$$V_{out} = B_{out} + \sum_{n=1}^{\infty}\left[\frac{D_n}{r^n}\cos n\phi - \frac{C_n}{r^n}\sin n\phi\right] \tag{0.3}$$

From the boundary condition requiring the potential to be continuous at the surface of the cylinder we get the relations

$$\begin{aligned} B_{out} &= B_{in} && (0.4)\\ C_n &= -A_n R^{2n} && (0.5)\\ D_n &= B_n R^{2n} && (0.6)\end{aligned}$$

where R is the radius of the cylinder.

From the condition on the derivative of the potential at the boundary, we get

$$\sum_{n=1}^{\infty}\left[2nR^{n-1}A_n\right]\sin n\phi+\sum_{n=1}^{\infty}\left[2nR^{n-1}B_n\right]\cos n\phi=\frac{\sigma_b}{\varepsilon_0} \tag{0.7}$$

From 0.1 we see that all coefficients of the sine terms are zero, as are all coefficients of cosine terms except for $n=1$. We therefore get

$$B_1 = \frac{P}{2\varepsilon_0} \tag{0.8}$$

$$D_1 = R^2B_1 \tag{0.9}$$

$$= \frac{PR^2}{2\varepsilon_0} \tag{0.10}$$

The potential in the two regions is thus

$$V_{in} = \frac{P}{2\varepsilon_0}r\cos\phi \tag{0.11}$$

$$V_{out} = \frac{PR^2}{2r\varepsilon_0}\cos\phi \tag{0.12}$$

From this we can get the field by taking the negative gradient. Since $r\cos\phi=x$, the field inside the cylinder is

$$\mathbf{E}_{in} = -\nabla V_{in} \tag{0.13}$$

$$= -\frac{P}{2\varepsilon_0}\hat{\mathbf{x}} \tag{0.14}$$

The interior field is thus uniform, just as the field inside a uniformly polarized sphere is uniform.

Outside the cylinder we need to take the gradient in cylindrical coordiantes. We get

$$\mathbf{E}_{out} = -\frac{\partial V_{out}}{\partial r}\hat{\mathbf{r}}-\frac{1}{r}\frac{\partial V_{out}}{\partial\phi}\hat{\phi} \tag{0.15}$$

$$= \frac{R^2}{2r^2\varepsilon_0}\left[P\cos\phi\hat{\mathbf{r}}+P\sin\phi\hat{\phi}\right] \tag{0.16}$$

We can write this in terms of the polarization vector. If $\mathbf{P}$ points in the direction $\phi=0$ then we have

$$\mathbf{P}=P\cos\phi\hat{\mathbf{r}}-P\sin\phi\hat{\phi} \tag{0.17}$$

where the minus sign on the second term is because $\hat{\phi}$ points in the direction of increasing ϕ, which is clockwise from $\phi = 0$. We have then

$$P\cos\phi\,\hat{\mathbf{r}} + P\sin\phi\,\hat{\phi} = 2\left(\mathbf{P}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}} - \mathbf{P} \tag{0.18}$$

so the field is

$$\mathbf{E}_{out} = \frac{R^2}{2r^2\varepsilon_0}\left[2\left(\mathbf{P}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}} - \mathbf{P}\right] \tag{0.19}$$