FIELD OF A POLARIZED CYLINDER

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 4.13.

As another example of the use of the bound charges representation of a polarized object, we can look at a cylinder which contains a uniform polarization $\mathbf{P}$ perpendicular to its axis. In this case, $\nabla\cdot\mathbf{P}=0$ everywhere inside the cylinder. If we take the direction of $\mathbf{P}$ to be $\phi=0$ then $\mathbf{P}\cdot\hat{\mathbf{n}}=P\cos\phi$. We thus must find the potential of an infinite cylinder with a surface charge of

$$\sigma_b = P\cos\phi \tag{1}$$

We've worked out the general solution to Laplace's equation in cylindrical coordinates before, so we can use the results from there. The solution inside the cylinder is

$$V_{in} = B_{in} + \sum_{n=1}^{\infty}\left[A_n r^n \sin n\phi + B_n r^n \cos n\phi\right] \tag{2}$$

while outside it is

$$V_{out} = B_{out} + \sum_{n=1}^{\infty}\left[\frac{D_n}{r^n}\cos n\phi - \frac{C_n}{r^n}\sin n\phi\right] \tag{3}$$

From the boundary condition requiring the potential to be continuous at the surface of the cylinder we get the relations

$$\begin{aligned} B_{out} &= B_{in} && (4)\\ C_n &= -A_n R^{2n} && (5)\\ D_n &= B_n R^{2n} && (6) \end{aligned}$$

where R is the radius of the cylinder.

From the condition on the derivative of the potential at the boundary, we get

$$\sum_{n=1}^{\infty}\left[2nR^{n-1}A_n\right]\sin n\phi + \sum_{n=1}^{\infty}\left[2nR^{n-1}B_n\right]\cos n\phi = \frac{\sigma_b}{\epsilon_0} \tag{7}$$

From 1 we see that all coefficients of the sine terms are zero, as are all coefficients of cosine terms except for $n=1$. We therefore get

$$\begin{aligned} B_1 &= \frac{P}{2\epsilon_0} && (8) \\ D_1 &= R^2 B_1 && (9) \\ &= \frac{PR^2}{2\epsilon_0} && (10) \end{aligned}$$

The potential in the two regions is thus

$$\begin{aligned} V_{in} &= \frac{P}{2\epsilon_0} r\cos\phi && (11) \\ V_{out} &= \frac{PR^2}{2r\epsilon_0}\cos\phi && (12) \end{aligned}$$

From this we can get the field by taking the negative gradient. Since $r\cos\phi = x$, the field inside the cylinder is

$$\begin{aligned} \mathbf{E}_{in} &= -\nabla V_{in} && (13) \\ &= -\frac{P}{2\epsilon_0}\hat{\mathbf{x}} && (14) \end{aligned}$$

The interior field is thus uniform, just as the field inside a uniformly polarized sphere is uniform.

Outside the cylinder we need to take the gradient in cylindrical coordiantes. We get

$$\begin{aligned} \mathbf{E}_{out} &= -\frac{\partial V_{out}}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial V_{out}}{\partial \phi}\hat{\phi} && (15) \\ &= \frac{R^2}{2r^2\epsilon_0}\left[P\cos\phi\hat{\mathbf{r}} + P\sin\phi\hat{\phi}\right] && (16) \end{aligned}$$

We can write this in terms of the polarization vector. If $\mathbf{P}$ points in the direction $\phi = 0$ then we have

$$\mathbf{P} = P\cos\phi\hat{\mathbf{r}} - P\sin\phi\hat{\phi} \tag{17}$$

where the minus sign on the second term is because $\hat{\phi}$ points in the direction of increasing ϕ, which is clockwise from $\phi = 0$. We have then

$$P\cos\phi\hat{\mathbf{r}} + P\sin\phi\hat{\phi} = 2(\mathbf{P}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{P} \tag{18}$$

so the field is

$$\mathbf{E}_{out}=\frac{R^{2}}{2r^{2}\epsilon_{0}}\left[2\left(\mathbf{P}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}-\mathbf{P}\right] \tag{19}$$