DIELECTRIC EXAMPLES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Section 4.4.1 and Problem 4.19.

In a linear dielectric, we have a simple relationship between the displacement, the field and the polarization. We have

$$\mathbf{D} = \epsilon_0\mathbf{E}+\mathbf{P} \tag{1}$$
$$= \epsilon_0(1+\chi_e)\mathbf{E} \tag{2}$$
$$\equiv \epsilon\mathbf{E} \tag{3}$$

The quantity $1+\chi_e$ is called the *dielectric constant* (sometimes called the *relative permittivity*) of a material. If we use the symbol ϵ_r for this, we have

$$\epsilon_r = 1+\chi_e \tag{4}$$
$$= \frac{\epsilon}{\epsilon_0} \tag{5}$$

From the definition of displacement, we have that

$$\nabla\cdot\mathbf{D} = \rho_f \tag{6}$$

where ρ_f is the free charge (as opposed to bound charge) in the dielectric. In the case of a linear dielectric, we have for the curl:

$$\nabla\times\mathbf{D} = \epsilon_0\nabla\times\mathbf{E}+\nabla\times\mathbf{P} \tag{7}$$
$$= \nabla\times\mathbf{P} \tag{8}$$

since $\nabla\times\mathbf{E}=0$ in all electrostatic problems. Is the curl of the polarization zero in general? We might think that, for linear dielectrics, it has to be, since $\mathbf{P}$ is proportional to $\mathbf{E}$. However, it is the constant of proportionality $\epsilon_0\chi_e$ that messes things up, since if we have two adjacent linear dielectrics with different susceptibilities, the constant is different on either side of the boundary. The argument goes something like this:

Suppose we want to calculate the line integral of $\mathbf{P}$ along a rectangular path where the long sides of the rectangle lie on opposite sides of the boundary (and are parallel to the boundary) while the short sides of the rectangle are perpendicular to the boundary. We can consider an extreme case where one side of the boundary is a vacuum (so $\mathbf{P} = 0$), while the other is a dielectric with $\mathbf{P} \neq 0$. In the general case where $\mathbf{P}$ is not normal to the boundary, $\mathbf{P} \cdot d\mathbf{l} \neq 0$ within the dielectric, so the line integral won't be zero. By Stokes's theorem, the curl isn't zero either.

At this point, you might think: but what about the case of the electric field at the boundary of a conductor? We know that $\mathbf{E} = 0$ inside the conductor, and if the conductor has some surface charge, $\mathbf{E} \neq 0$ outside the conductor. Can't we use the same argument to show that $\nabla \times \mathbf{E} \neq 0$ in this case also? The key point here is that the electric field just outside a conductor is always normal to its surface, so in this case $\mathbf{E} \cdot d\mathbf{l} = 0$, so the line integral is zero and the curl is zero as it has to be. (Actually, we know from the original derivation of the $\nabla \times \mathbf{E} = 0$ condition, using the fact that the electric field is the gradient of a potential, that the line integral around *any* path is zero, so the argument doesn't rely on our choosing a rectangle as the path. The electric field will always be oriented in such a way that the line integral comes out to zero.)

Returning to the polarization, we can see that if the dielectric is uniform and homogeneous (that is, it's the same dielectric everywhere there is an electric field), then the curl *will* be zero, and in fact the displacement $\mathbf{D}$ satisfies (almost) the same equations as that of the vacuum electric field $\mathbf{E}_v$:

$$\nabla \cdot \mathbf{D} = \rho_f = \varepsilon_0 \nabla \cdot \mathbf{E}_v \tag{9}$$

$$\nabla \times \mathbf{D} = 0 = \nabla \times \mathbf{E}_v \tag{10}$$

Whenever two vector fields have the same divergence and curl, they must be equal, so we can say that

$$\mathbf{D} = \varepsilon_0 \mathbf{E}_v \tag{11}$$

However, we also know that

$$\mathbf{D} = \varepsilon \mathbf{E} \tag{12}$$

where the $\mathbf{E}$ in this relation is actual electric field inside the linear dielectric. Therefore

$$\mathbf{E} = \frac{\varepsilon_0}{\varepsilon}\mathbf{E}_v \tag{13}$$

$$= \frac{1}{\varepsilon_r}\mathbf{E}_v \tag{14}$$

That is, a uniform dielectric reduces the electric field by a factor equal to the dielectric constant.

As an example, suppose we have a parallel plate capacitor with a spacing d between the plates. Half the distance between the plates is filled by a dielectric slab with constant ε_r with the remaining half being a vacuum. If the potential difference between the plates in a vacuum is V, what will the potential difference be with the dielectric?

The half of the distance that is vacuum will be unaffected by the dielectric, and accounts for half the original potential difference, or $V/2$. In the dielectric the electric field is reduced by a factor ε_r, so the potential difference within the dielectric will be $V/2\varepsilon_r$. The total potential difference will now be

$$V_1 = \frac{V}{2}\left(1+\frac{1}{\varepsilon_r}\right) \tag{15}$$

$$= \frac{(1+\varepsilon_r)V}{2\varepsilon_r} \tag{16}$$

This effectively changes the capacitance, since $C = Q/V$. Thus the ratio of the new capacitance to the original is

$$\frac{C_1}{C} = \frac{V}{V_1} \tag{17}$$

$$= \frac{2}{(1+1/\varepsilon_r)} \tag{18}$$

$$= \frac{2\varepsilon_r}{1+\varepsilon_r} \tag{19}$$

The actual location of the dielectric slab doesn't matter in this calculation, but to make things definite, suppose it's positioned midway between the plates, so there is a gap of $d/4$ on each side of the slab. We can then use similar techniques to the earlier post to find the various quantities.

The electric field in the vacuum sections is

$$E_v = \frac{V/2}{d/2} \tag{20}$$

$$= \frac{V}{d} \tag{21}$$

$$= \frac{2\varepsilon_r}{1+\varepsilon_r}\frac{V_1}{d} \tag{22}$$

That is, it's unchanged from the pure vacuum case. Within the dielectric, we have

$$E_d = \frac{E_v}{\varepsilon_r} \tag{23}$$

$$= \frac{V}{\varepsilon_r d} \tag{24}$$

$$= \frac{2}{1+\varepsilon_r}\frac{V_1}{d} \tag{25}$$

The surface charge on the capacitor plates must produce the vacuum field, so

$$\sigma = \varepsilon_0 E_v \tag{26}$$

$$= \varepsilon_0 \frac{V}{d} \tag{27}$$

$$= \frac{2\varepsilon_r\varepsilon_0}{1+\varepsilon_r}\frac{V_1}{d} \tag{28}$$

The polarization in the vacuum is zero, and within the dielectric it is

$$P_d = \varepsilon_0 (\varepsilon_r - 1) E_d \tag{29}$$

$$= \varepsilon_0 \frac{\varepsilon_r - 1}{\varepsilon_r}\frac{V}{d} \tag{30}$$

$$= 2\varepsilon_0 \frac{\varepsilon_r - 1}{\varepsilon_r + 1}\frac{V_1}{d} \tag{31}$$

Since the bound surface charge on the dielectric is $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$ on the upper surface of the dielectric the polarization and normal vector point in opposite directions so

$$\sigma_b = -2\varepsilon_0 \frac{\varepsilon_r - 1}{\varepsilon_r + 1}\frac{V_1}{d} \tag{32}$$

Finally, the displacement is $D = \varepsilon_0 E + P$ (since all vectors are parallel here, we can use magnitudes), so

$$D_v = \frac{2\varepsilon_0\varepsilon_r}{1+\varepsilon_r}\frac{V_1}{d} \tag{33}$$

$$D_d = \varepsilon_0 E_d + P_d \tag{34}$$

$$= \frac{2\varepsilon_0\varepsilon_r}{1+\varepsilon_r}\frac{V_1}{d} \tag{35}$$

The displacement is the same everywhere between the plates.

Now suppose the dielectric covers the entire gap between the plates, but occupies only half the area of the plates. Because the capacitor plates are conductors, the potential must be the same everywhere inside a given plate. Therefore, the potential difference is the same whether we go through the dielectric or the vacuum. This in turn means the electric field must be the same on both sides, and since the electric field inside a dielectric is reduced by a factor ε_r, there must be more charge on the plate where it contacts the dielectric. That is, on the vacuum side we have

$$E_v = \frac{V_2}{d} \tag{36}$$

$$= \frac{\sigma_v}{\varepsilon_0} \tag{37}$$

while on the dielectric side we have

$$E_d = E_v \tag{38}$$

$$= \frac{\sigma_d}{\varepsilon_0} \tag{39}$$

$$= \frac{\varepsilon_r\sigma_v}{\varepsilon_0} \tag{40}$$

So the charge density on the plate next to the dielectric is

$$\sigma_d = \varepsilon_r\sigma_v \tag{41}$$

$$= \varepsilon_r\varepsilon_0\frac{V_2}{d} \tag{42}$$

This means that the capacitor holds more charge than if there were a vacuum between its plates, so, since both regions contribute an equal amount:

$$\frac{C_2}{C} = \frac{\sigma_v + \sigma_d}{2\sigma_v} \tag{43}$$

$$= \frac{1+\varepsilon_r}{2} \tag{44}$$

The polarization within the dielectric is

$$P_d = \varepsilon_0(\varepsilon_r - 1)E_d \tag{45}$$

$$= \varepsilon_0\left(\varepsilon_r - 1\right)\frac{V_2}{d} \tag{46}$$

The surface charge on the dielectric is

$$\sigma_b = -\varepsilon_0\left(\varepsilon_r - 1\right)\frac{V_2}{d} \tag{47}$$

The displacement is

$$D_v = \varepsilon_0 E_v \tag{48}$$

$$= \varepsilon_0 \frac{V_2}{d} \tag{49}$$

$$D_d = \varepsilon_0 E_d + P_d \tag{50}$$

$$= \varepsilon_0 \frac{V_2}{d} + \varepsilon_0\left(\varepsilon_r - 1\right)\frac{V_2}{d} \tag{51}$$

$$= \varepsilon_0 \varepsilon_r \frac{V_2}{d} \tag{52}$$

PINGBACKS

Pingback: Point charge embedded in dielectric plane