DIELECTRIC SPHERE WITH FREE CHARGE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Section 4.4.1 and Problem 4.20.

Suppose we have a sphere of linear dielectric (radius R) in which is embedded a uniform free charge of density ρ.

From the definition of displacement, we have that

$$\nabla\cdot\mathbf{D}=\rho \tag{1}$$

The integral form of this is, using Gauss's law:

$$\int\mathbf{D}\cdot d\mathbf{a}=Q \tag{2}$$

where Q is the charge enclosed by the surface over which the integral is done. In this case, from symmetry, we get for points $r<R$:

$$\begin{aligned} 4\pi r^2 D &= \frac{4}{3}\pi r^3\rho & (3)\\ D &= \frac{r}{3}\rho & (4) \end{aligned}$$

Outside the sphere, we have

$$\begin{aligned} 4\pi r^2 D &= \frac{4}{3}\pi R^3\rho & (5)\\ D &= \frac{R^3}{3r^2}\rho & (6) \end{aligned}$$

In both regions, the relation between D and E is $D=\epsilon_0\epsilon_r E$ where ϵ_r is the dielectric constant, so we have

$$E=\begin{cases}\frac{r\rho}{3\epsilon_0\epsilon_r} & 0<r<R\\ \frac{R^3\rho}{3\epsilon_0 r^2} & r>R\end{cases} \tag{7}$$

Outside the sphere the dielectric constant is 1, since we are in a vacuum there.

From this we can find the potential at the centre of the sphere, relative to infinity.

$$V = -\int_{\infty}^{0} E dr \tag{8}$$

$$= -\frac{R^3\rho}{3\epsilon_0}\int_{\infty}^{R}\frac{dr}{r^2} - \frac{\rho}{3\epsilon_0\epsilon_r}\int_{R}^{0} r dr \tag{9}$$

$$= \frac{R^2\rho}{3\epsilon_0} + \frac{R^2\rho}{6\epsilon_0\epsilon_r} \tag{10}$$

$$= \frac{R^2\rho}{3\epsilon_0}\left[1 + \frac{1}{2\epsilon_r}\right] \tag{11}$$