DIELECTRIC SPHERE WITH FREE CHARGE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Section 4.4.1 and Problem 4.20.

Suppose we have a sphere of linear dielectric (radius R) in which is embedded a uniform free charge of density ρ.

From the definition of displacement, we have that

$$\nabla\cdot\mathbf{D}=\rho \tag{0.1}$$

The integral form of this is, using Gauss's law:

$$\int\mathbf{D}\cdot d\mathbf{a}=Q \tag{0.2}$$

where Q is the charge enclosed by the surface over which the integral is done. In this case, from symmetry, we get for points $r<R$:

$$4\pi r^{2}D = \frac{4}{3}\pi r^{3}\rho \tag{0.3}$$

$$D = \frac{r}{3}\rho \tag{0.4}$$

Outside the sphere, we have

$$4\pi r^{2}D = \frac{4}{3}\pi R^{3}\rho \tag{0.5}$$

$$D = \frac{R^{3}}{3r^{2}}\rho \tag{0.6}$$

In both regions, the relation between D and E is $D=\varepsilon_0\varepsilon_r E$ where ε_r is the dielectric constant, so we have

$$E=\begin{cases}\frac{r\rho}{3\varepsilon_0\varepsilon_r} & 0<r<R\\ \frac{R^3\rho}{3\varepsilon_0 r^2} & r>R\end{cases} \tag{0.7}$$

Outside the sphere the dielectric constant is 1, since we are in a vacuum there.

From this we can find the potential at the centre of the sphere, relative to infinity.

$$V = -\int_{\infty}^{0} E dr \tag{0.8}$$

$$= -\frac{R^3\rho}{3\varepsilon_0}\int_{\infty}^{R}\frac{dr}{r^2} - \frac{\rho}{3\varepsilon_0\varepsilon_r}\int_{R}^{0} r dr \tag{0.9}$$

$$= \frac{R^2\rho}{3\varepsilon_0} + \frac{R^2\rho}{6\varepsilon_0\varepsilon_r} \tag{0.10}$$

$$= \frac{R^2\rho}{3\varepsilon_0}\left[1 + \frac{1}{2\varepsilon_r}\right] \tag{0.11}$$