DIELECTRIC CYLINDER IN UNIFORM ELECTRIC FIELD

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 4.22.

In our original investigation into electrostatic boundary conditions in a vacuum, we found that the electric field is discontinuous across a layer of surface charge. The discontinuity was

$$E_{\perp}^{above} - E_{\perp}^{below} = \frac{\sigma}{\epsilon_0} \tag{1}$$

We can do a similar analysis for a dielectric by using the displacement instead of the bare electric field. We know that

$$\int_A \mathbf{D} \cdot d\mathbf{a} = Q_f \tag{2}$$

where Q_f is the free charge enclosed by the surface of integration. Therefore, if we consider a dielectric with a free surface charge of σ_f, we can use a small cylindrical Gaussian surface with ends on either side of the surface to say that

$$D_{\perp}^{above} - D_{\perp}^{below} = \sigma_f \tag{3}$$

Thus if there is no *free* surface charge on the dielectric, the perpendicular component of **D** is continuous across the surface. The dielectric may be polarized, since polarization induces only a bound surface charge, and that has no effect on the continuity of $D_{\perp}$.

In a linear dielectric, $\mathbf{D} = \epsilon \mathbf{E}$ and as usual $\mathbf{E} = -\nabla V$, so we can convert this into a condition on the potential on each side of the surface layer:

$$\epsilon_1 \frac{\partial V_1}{\partial n} - \epsilon_2 \frac{\partial V_2}{\partial n} = \sigma_f \tag{4}$$

As an example, suppose we have a long cylinder of dielectric that is in a uniform electric field $\mathbf{E}_0$, where the field is perpendicular to the axis of the cylinder. We can find the potential everywhere, and thus find the field inside the cylinder.

To solve this problem, we set up a cylindrical coordinate system with its axis centred on the axis of the cylinder. We can define the $\phi = 0$ direction as the direction of the electric field. In addition to the boundary condition 4, we have the addition boundary condition that the potential must be continuous, and we also have the asymptotic condition that for large r, the electric field must tend to a constant. Since there is no free charge, $\sigma_f = 0$ and the three conditions become

$$\epsilon_1 \left.\frac{\partial V_1}{\partial n}\right|_{r=R} = \epsilon_2 \left.\frac{\partial V_2}{\partial n}\right|_{r=R} \tag{5}$$

$$V_1(R) = V_2(R) \tag{6}$$

$$\lim_{r\to\infty} V_2(r) = -E_0 r\cos\phi \tag{7}$$

Here, V_1 is the potential inside the cylinder and V_2 is the potential outside. R is the radius of the cylinder. The last condition says that the potential for large r must be $-E_0\zeta$ where $\zeta \equiv r\cos\phi$ is a coordinate measured along the $\phi = 0$ direction. It would be confusing to call this coordinate z, since in cylindrical coordinates, z denotes the direction along the axis of the cylinder, which we've already said is the axis of the dielectric. With this definition, the field is $\mathbf{E}_0 = -(dV/d\zeta)\hat{\zeta} = E_0\hat{\zeta}$.

Since there is no free charge, the system satisfies Laplace's equation, so we can make use of our solution in cylindrical coordinates. The general solution is

$$V(r,\phi) = A\ln r + K + \sum_{n=1}^{\infty} r^n (A_n \sin n\phi + B_n \cos n\phi) + \sum_{n=1}^{\infty} \frac{1}{r^n}(C_n \sin n\phi + D_n \cos n\phi) \tag{8}$$

Inside the cylinder we can throw away the log term and the $1/r^n$ terms to keep the potential finite at $r = 0$. Thus inside we have

$$V_1 = K_1 + \sum_{n=1}^{\infty} r^n (A_n \sin n\phi + B_n \cos n\phi) \tag{9}$$

Outside, we again throw away the log term. We can also throw away all the r^n terms *except* for $n = 1$, since we need the asymptotic behaviour referred to above. Thus we get

$$V_2 = K_2 - E_0 r\cos\phi + \sum_{n=1}^{\infty} \frac{1}{r^n}(C_n \sin n\phi + D_n \cos n\phi) \tag{10}$$

We can now apply the two other boundary conditions above. First:

$$V_1(R) = V_2(R) \tag{11}$$

$$K_1 + \sum_{n=1}^{\infty} R^n (A_n \sin n\phi + B_n \cos n\phi) = K_2 - E_0 R \cos\phi + \sum_{n=1}^{\infty} \frac{1}{R^n} (C_n \sin n\phi + D_n \cos n\phi) \tag{12}$$

From the uniqueness of series, we can equate coefficients of the various trig functions, so we get

$$\begin{aligned} K_1 &= K_2 && (13)\\ A_n R^n &= \frac{C_n}{R^n} && (14)\\ B_n R^n &= \frac{D_n}{R^n} \qquad (n \neq 1) && (15)\\ R B_1 &= -E_0 R + \frac{D_1}{R} && (16) \end{aligned}$$

Since the value of the constant K_1 makes no difference to the field (it disappears when we take the derivative), we might as well take $K_1 = K_2 = 0$.

Now from the derivative condition, we have, using $\epsilon_1 = \epsilon_0 \epsilon_r$, where ϵ_r is the dielectric constant, and $\epsilon_2 = \epsilon_0$, since outside the dielectric we have a vacuum:

$$\epsilon_1 \left.\frac{\partial V_1}{\partial n}\right|_{r=R} = \epsilon_2 \left.\frac{\partial V_2}{\partial n}\right|_{r=R} \tag{17}$$

$$\epsilon_r \left[\sum_{n=1}^{\infty} n R^{n-1} (A_n \sin n\phi + B_n \cos n\phi)\right] = -E_0 \cos\phi - \sum_{n=1}^{\infty} \frac{n}{R^{n+1}} (C_n \sin n\phi + D_n \cos n\phi) \tag{18}$$

Again, equating coefficients, we get

$$\begin{aligned} \epsilon_r n A_n R^{n-1} &= -\frac{n}{R^{n+1}} C_n && (19)\\ \epsilon_r n B_n R^{n-1} &= -\frac{n}{R^{n+1}} D_n \qquad (n \neq 1) && (20)\\ \epsilon_r B_1 &= -E_0 - \frac{D_1}{R^2} && (21) \end{aligned}$$

Combining these equations with those from the first boundary condition above, we get, in each of the first two equations, the condition that either $\epsilon_r = -1$ (impossible, since dielectric constants are always ≥ 1) or

$A_n = C_n = 0$ and, for $n \neq 1$, $B_n = D_n = 0$. We are therefore left with the last equation from each boundary condition, and we can solve these two equations to get

$$B_1 = -\frac{2E_0}{\epsilon_r + 1} \tag{22}$$

$$D_1 = \frac{\epsilon_r - 1}{\epsilon_r + 1} R^2 E_0 \tag{23}$$

We therefore get for the potentials

$$V_1 = -\frac{2E_0}{\epsilon_r + 1} r\cos\phi \tag{24}$$

$$V_2 = \frac{\epsilon_r - 1}{\epsilon_r + 1} \frac{R^2 E_0}{r} \cos\phi - E_0 r\cos\phi \tag{25}$$

From this we can get the field inside the cylinder

$$V_1 = -\frac{2E_0}{\epsilon_r + 1} \zeta \tag{26}$$

$$\mathbf{E}_{in} = -\frac{dV_1}{d\zeta} \hat{\zeta} \tag{27}$$

$$= \frac{2}{\epsilon_r + 1} \mathbf{E}_0 \tag{28}$$

Rather surprisingly, the field inside the cylinder is uniform.

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