

DIELECTRIC SHELL SURROUNDING CONDUCTING SPHERE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 4.24.

As an exercise in applying Laplace's equation to a problem with dielectrics, suppose we have a conducting sphere of radius a surrounded by a spherical shell of dielectric of outer radius b and susceptibility ϵ , with the whole system in a uniform electric field \mathbf{E}_0 .

The general solution to Laplace's equation for the potential in spherical coordinates is

$$(0.1) \quad V(r, \theta) = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos \theta)$$

where P_l is the degree- l Legendre polynomial.

Inside the sphere, the potential is constant (since the field is zero inside a conductor), so we might as well take it to be zero.

In the region $a < r < b$, the solution is the general one given above. For $r > b$, in order to avoid an infinite field for large r , we have to drop the A_l terms except for A_1 since we need a potential that gives a constant field. If we take the field to lie in the z direction, then since $z = r \cos \theta = r P_1(\cos \theta)$ we have for this region

$$(0.2) \quad V_{r>b} = -E_0 r P_1 + \sum_{n=0}^{\infty} \frac{C_n}{r^{n+1}} P_n(\cos \theta)$$

Note that we've used a different set of coefficients C_n here, since the solution in this region is distinct from that for the region $a < r < b$. That is, the coefficients $B_n \neq C_n$.

We can obtain conditions on the coefficients by equating terms of the various Legendre polynomials in the boundary conditions. Continuity of the potential at $r = a$ gives the condition

$$(0.3) \quad A_n a^n + \frac{B_n}{a^{n+1}} = 0$$

Continuity at $r = b$ gives

$$(0.4) \quad A_1 b + \frac{B_1}{b^2} = -E_0 b + \frac{C_1}{b^2}$$

$$(0.5) \quad A_n b^n + \frac{B_n}{b^{n+1}} = \frac{C_n}{b^{n+1}} \quad (n \neq 1)$$

Finally, we can use the condition at the boundary of two dielectrics to get, since there is no free charge at the boundary:

$$(0.6) \quad \epsilon_1 \frac{\partial V_1}{\partial n} = \epsilon_2 \frac{\partial V_2}{\partial n}$$

$$(0.7) \quad \epsilon A_n n b^{n-1} - \epsilon \frac{(n+1)B_n}{b^{n+2}} = -\epsilon_0 \frac{(n+1)C_n}{b^{n+2}} \quad (n \neq 1)$$

$$(0.8) \quad \epsilon A_1 - 2\epsilon \frac{B_1}{b^3} = -\epsilon_0 E_0 - 2\epsilon_0 \frac{C_1}{b^3}$$

Consider first the terms for $n \neq 1$. We get

$$(0.9) \quad B_n = -a^{2n+1} A_n$$

$$(0.10) \quad A_n \left(b^n - \frac{a^{2n+1}}{b^{n+1}} \right) = \frac{C_n}{b^{n+1}}$$

$$(0.11) \quad C_n = A_n (b^{2n+1} - a^{2n+1})$$

Since both B_n and C_n are proportional to A_n , the only way the third boundary condition (at the dielectric/vacuum boundary) above can be satisfied is if $A_n = B_n = C_n = 0$ for $n \neq 1$. We are therefore left with the $n = 1$ terms. For these we get

$$(0.12) \quad B_1 = -a^3 A_1$$

$$(0.13) \quad A_1 \left(b - \frac{a^3}{b^2} \right) = -E_0 b + \frac{C_1}{b^2}$$

$$(0.14) \quad C_1 = A_1 (b^3 - a^3) + b^3 E_0$$

Plugging these into the third boundary condition gives

$$(0.15) \quad A_1 = \frac{-3\epsilon_0 b^3 E_0}{\epsilon (b^3 + 2a^3) + 2\epsilon_0 (b^3 - a^3)}$$

$$(0.16) \quad B_1 = \frac{3\epsilon_0 a^3 b^3 E_0}{\epsilon (b^3 + 2a^3) + 2\epsilon_0 (b^3 - a^3)}$$

In terms of the dielectric constant $\epsilon_r = \epsilon/\epsilon_0$ we get

$$(0.17) \quad A_1 = \frac{-3b^3 E_0}{\epsilon_r (b^3 + 2a^3) + 2(b^3 - a^3)}$$

$$(0.18) \quad B_1 = \frac{3a^3 b^3 E_0}{\epsilon_r (b^3 + 2a^3) + 2(b^3 - a^3)}$$

The potential inside the dielectric shell is therefore

$$(0.19) \quad V_{a < r < b}(r, \theta) = A_1 r \cos \theta + \frac{B_1}{r^2} \cos \theta$$

$$(0.20) \quad = \frac{3b^3 E_0 \cos \theta}{\epsilon_r (b^3 + 2a^3) + 2(b^3 - a^3)} \left[-r + \frac{a^3}{r^2} \right]$$

The field can be found from the gradient

$$(0.21)$$

$$\mathbf{E} = -\nabla V$$

$$(0.22)$$

$$= -\frac{\partial V}{\partial r} \hat{\mathbf{r}} - \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\boldsymbol{\theta}}$$

$$(0.23)$$

$$= \frac{3b^3 E_0}{\epsilon_r (b^3 + 2a^3) + 2(b^3 - a^3)} \left[\left(1 + 2\frac{a^3}{r^3} \right) \cos \theta \hat{\mathbf{r}} + \left(-1 + \frac{a^3}{r^3} \right) \sin \theta \hat{\boldsymbol{\theta}} \right]$$

This reduces to the situation of a conducting sphere in a uniform field if we set $\epsilon_r = 1$ (effectively replacing the dielectric by a vacuum). In that case, the potential reduces to

$$(0.24) \quad V_{\epsilon_r=1} = -E_0 r \cos \theta + \frac{a^3}{r^2} E_0 \cos \theta$$

The first term is just the applied field, and the second term arises from the induced charge on the conductor.