DIELECTRIC SHELL SURROUNDING CONDUCTING SPHERE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 4.24.

As an exercise in applying Laplace's equation to a problem with dielectrics, suppose we have a conducting sphere of radius a surrounded by a spherical shell of dielectric of outer radius b and susceptibility ε, with the whole system in a uniform electric field $\mathbf{E}_0$.

The general solution to Laplace's equation for the potential in spherical coordinates is

$$V(r,\theta)=\sum_{n=0}^{\infty}\left(A_n r^n+\frac{B_n}{r^{n+1}}\right)P_n(\cos\theta) \tag{1}$$

where P_l is the degree-l Legendre polynomial.

Inside the sphere, the potential is constant (since the field is zero inside a conductor), so we might as well take it to be zero.

In the region $a<r<b$, the solution is the general one given above. For $r>b$, in order to avoid an infinite field for large r, we have to drop the A_l terms except for A_1 since we need a potential that gives a constant field. If we take the field to lie in the z direction, then since $z=r\cos\theta=rP_1(\cos\theta)$ we have for this region

$$V_{r>b}=-E_0 rP_1+\sum_{n=0}^{\infty}\frac{C_n}{r^{n+1}}P_n(\cos\theta) \tag{2}$$

Note that we've used a different set of coefficients C_n here, since the solution in this region is distinct from that for the region $a<r<b$. That is, the coefficients $B_n\neq C_n$.

We can obtain conditions on the coefficients by equating terms of the various Legendre polynomials in the boundary conditions. Continuity of the potential at $r=a$ gives the condition

$$A_n a^n+\frac{B_n}{a^{n+1}}=0 \tag{3}$$

Continuity at $r=b$ gives

$$A_1 b+\frac{B_1}{b^2} = -E_0 b+\frac{C_1}{b^2} \tag{4}$$

$$A_n b^n+\frac{B_n}{b^{n+1}} = \frac{C_n}{b^{n+1}} \qquad (n\neq 1) \tag{5}$$

Finally, we can use the condition at the boundary of two dielectrics to get, since there is no free charge at the boundary:

$$\varepsilon_1\frac{\partial V_1}{\partial n}=\varepsilon_2\frac{\partial V_2}{\partial n} \tag{6}$$

$$\varepsilon A_n n b^{n-1}-\varepsilon\frac{(n+1)B_n}{b^{n+2}}=-\varepsilon_0\frac{(n+1)C_n}{b^{n+2}} \qquad (n\neq 1) \tag{7}$$

$$\varepsilon A_1-2\varepsilon\frac{B_1}{b^3}=-\varepsilon_0 E_0-2\varepsilon_0\frac{C_1}{b^3} \tag{8}$$

Consider first the terms for $n\neq 1$. We get

$$B_n = -a^{2n+1}A_n \tag{9}$$

$$A_n\left(b^n-\frac{a^{2n+1}}{b^{n+1}}\right) = \frac{C_n}{b^{n+1}} \tag{10}$$

$$C_n = A_n\left(b^{2n+1}-a^{2n+1}\right) \tag{11}$$

Since both B_n and C_n are proportional to A_n, the only way the third boundary condition (at the dielectric/vacuum boundary) above can be satisfied is if $A_n=B_n=C_n=0$ for $n\neq 1$. We are therefore left with the $n=1$ terms. For these we get

$$B_1 = -a^3 A_1 \tag{12}$$

$$A_1\left(b-\frac{a^3}{b^2}\right) = -E_0 b+\frac{C_1}{b^2} \tag{13}$$

$$C_1 = A_1\left(b^3-a^3\right)+b^3E_0 \tag{14}$$

Plugging these into the third boundary condition gives

$$A_1 = \frac{-3\varepsilon_0 b^3 E_0}{\varepsilon\left(b^3+2a^3\right)+2\varepsilon_0\left(b^3-a^3\right)} \tag{15}$$

$$B_1 = \frac{3\varepsilon_0 a^3 b^3 E_0}{\varepsilon\left(b^3+2a^3\right)+2\varepsilon_0\left(b^3-a^3\right)} \tag{16}$$

In terms of the dielectric constant $\varepsilon_r=\varepsilon/\varepsilon_0$ we get

$$A_1 = \frac{-3b^3E_0}{\varepsilon_r\left(b^3+2a^3\right)+2\left(b^3-a^3\right)} \tag{17}$$

$$B_1 = \frac{3a^3b^3E_0}{\varepsilon_r\left(b^3+2a^3\right)+2\left(b^3-a^3\right)} \tag{18}$$

The potential inside the dielectric shell is therefore

$$V_{a<r<b}(r,\theta) = A_1 r\cos\theta + \frac{B_1}{r^2}\cos\theta \tag{19}$$

$$= \frac{3b^3E_0\cos\theta}{\varepsilon_r\left(b^3+2a^3\right)+2\left(b^3-a^3\right)}\left[-r+\frac{a^3}{r^2}\right] \tag{20}$$

The field can be found from the gradient

$$\mathbf{E} = -\nabla V \tag{21}$$

$$= -\frac{\partial V}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} \tag{22}$$

$$= \frac{3b^3E_0}{\varepsilon_r\left(b^3+2a^3\right)+2\left(b^3-a^3\right)}\left[\left(1+2\frac{a^3}{r^3}\right)\cos\theta\,\hat{\mathbf{r}} + \left(-1+\frac{a^3}{r^3}\right)\sin\theta\,\hat{\theta}\right] \tag{23}$$

This reduces to the situation of a conducting sphere in a uniform field if we set $\varepsilon_r = 1$ (effectively replacing the dielectric by a vacuum). In that case, the potential reduces to

$$V_{\varepsilon_r=1} = -E_0 r\cos\theta + \frac{a^3}{r^2}E_0\cos\theta \tag{24}$$

The first term is just the applied field, and the second term arises from the induced charge on the conductor.