POINT CHARGE EMBEDDED IN DIELECTRIC PLANE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 4.25.

The following is a problem that combines the theory of linear dielectrics with the method of images. Suppose we divide 3-d space into two parts, separated by the plane $z=0$. The region $z<0$ is filled with a dielectric with constant ε_r, while the region $z>0$ is filled with a dielectric with a different constant ε_r'. We also place a point charge q on the z axis at $z=d$.

This problem is similar to that of a point charge above a conducting plane that served as the introductory example for the method of images. However, in that case, we were able to replace the conductor by a single point charge $-q$ at location $z=-d$ and then show that the resulting potential was valid for $z>0$. (Inside a grounded conductor, $V=0$.) Here, the situation isn't quite as simple.

The point charge will polarize both dielectrics with the result that there will be bound surface charge where the dielectrics meet. We call this surface charge σ_b and σ_b' for $z<0$ and $z>0$ respectively. The surface charge is related to the polarization by

$$\sigma_b=\mathbf{P}\cdot\hat{\mathbf{n}} \tag{1}$$

and for a linear dielectric

$$\mathbf{P}=\varepsilon_0\chi_e\mathbf{E} \tag{2}$$

so at the boundary between the dielectrics, we get

$$\begin{aligned}
\sigma_b &= \mathbf{P}\cdot\hat{\mathbf{n}} && (3)\\
&= \varepsilon_0\chi_e E_z && (4)\\
\sigma_b' &= \mathbf{P}'\cdot\hat{\mathbf{n}}' && (5)\\
&= -\varepsilon_0\chi_e' E_z' && (6)
\end{aligned}$$

We need to be careful with the directions of the various vectors here. Suppose that q is positive. Then the induced polarization will be such that negative charges are attracted to q, so that σ_b will consist of negative charge and σ_b' will consist of positive charge. That is, the polarization vector points towards q in both cases.

In the calculation of $\mathbf{P}\cdot\hat{\mathbf{n}}$ the normal vector to the plane points upwards but in the calculation of $\mathbf{P}'\cdot\hat{\mathbf{n}}'$ the normal vector points *downwards*, while $\mathbf{P}'$ still points in the same direction as $\mathbf{P}$. That is the reason for the minus sign in the result for σ_b'.

The electric field is discontinuous across a layer of surface charge, with the discontinuity given by

$$E_{z(above)}-E_{z(below)}=\frac{\sigma}{\varepsilon_0} \tag{7}$$

The fields due to the surface charges in this problem are therefore

$$\begin{aligned}
E_z' &= E_{z(above)} && (8)\\
&= \frac{\sigma_b+\sigma_b'}{2\varepsilon_0} && (9)\\
E_z &= E_{z(below)} && (10)\\
&= -\frac{\sigma_b+\sigma_b'}{2\varepsilon_0} && (11)
\end{aligned}$$

The perpendicular component of the field due to the point charge is, at $z=0$:

$$\begin{aligned}
E_{z(q)} &= -\frac{1}{4\pi\varepsilon'}\frac{q}{r^2+d^2}\cos\theta && (12)\\
&\;-\frac{1}{4\pi\varepsilon'}\frac{qd}{\left(r^2+d^2\right)^{3/2}} && (13)
\end{aligned}$$

Here $\varepsilon'=\varepsilon_0\varepsilon_r$ and reflects the fact that the electric field is reduced by the factor of ε_r inside a dielectric. The variable r is the distance from the origin to a point on the $z=0$ plane, as usual. The angle θ is that between the z axis and a line from q to the point on the plane, so that $\cos\theta=d/\sqrt{d^2+r^2}$. This formula is valid on both sides of the boundary.

We now have enough information to write equations for the two surface charges. We get

$$\sigma_b = \varepsilon_0 \chi_e E_z \tag{14}$$

$$= \varepsilon_0 \chi_e \left[-\frac{1}{4\pi\varepsilon'} \frac{qd}{\left(r^2+d^2\right)^{3/2}} - \frac{\sigma_b + \sigma_b'}{2\varepsilon_0} \right] \tag{15}$$

$$\sigma_b' = -\varepsilon_0 \chi_e' E_z' \tag{16}$$

$$= \varepsilon_0 \chi_e' \left[\frac{1}{4\pi\varepsilon'} \frac{qd}{\left(r^2+d^2\right)^{3/2}} - \frac{\sigma_b + \sigma_b'}{2\varepsilon_0} \right] \tag{17}$$

As a check at this stage, we observe that if we set $\chi_e = \chi_e'$ (that is, we make the same dielectric fill all space, thus eliminating the boundary), and then add these two equations together, we get

$$\sigma_b + \sigma_b' = -\chi_e \left(\sigma_b + \sigma_b'\right) \tag{18}$$

Since $\chi_e \neq 0$ in general, this means that $\sigma_b + \sigma_b' = 0$ in this case. This makes sense, since with no boundary between the dielectrics, we would expect there to be no net surface charge.

Returning to the general case, we can solve these two simultaneous equations (by hand or using Maple), and get, using the relation $\varepsilon' = \varepsilon_0 \left(1 + \chi_e'\right)$

$$\sigma_b = -\frac{\chi_e q d}{2\pi \left(2+\chi_e+\chi_e'\right)\left(r^2+d^2\right)^{3/2}} \tag{19}$$

$$\sigma_b' = \frac{\chi_e' q d}{2\pi \left(2+\chi_e+\chi_e'\right)\left(r^2+d^2\right)^{3/2}} \frac{\left(1+\chi_e\right)}{\left(1+\chi_e'\right)} \tag{20}$$

Again, we note that if $\chi_e = \chi_e'$, $\sigma_b = -\sigma_b'$.

In general:

$$\sigma_b + \sigma_b' = \frac{\left(\chi_e' - \chi_e\right) q d}{2\pi \left(2+\chi_e+\chi_e'\right)\left(1+\chi_e'\right)\left(r^2+d^2\right)^{3/2}} \tag{21}$$

At this stage, if we want to find the potential, we might try a direct integration, since

$$4\pi\varepsilon_0 V(\mathbf{r}) = \frac{q}{|\mathbf{r}-\mathbf{d}|} + \int \frac{\sigma_b(\mathbf{r}') + \sigma_b'(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dx' dy' \tag{22}$$

where $\mathbf{r}$ is the point at which we wish to find the potential, and the integration variable $\mathbf{r}'$ ranges over the plane $z = 0$. However, I couldn't find any way of doing this integral (Maple just got stuck) V.

I've never liked the method of images since it's largely informed guesswork, but anyway...

If we work out the total bound surface charge we get

$$Q = 2\pi\int_0^\infty \left(\sigma_b+\sigma_b'\right) r dr \tag{23}$$

$$= \frac{\left(\chi_e'-\chi_e\right)q}{\left(1+\chi_e'\right)\left(2+\chi_e+\chi_e'\right)} \tag{24}$$

We pause here to look at a few limiting cases. If $\chi_e'=0$, so that the upper region becomes a vacuum, we get

$$Q=\frac{-\chi_e}{2+\chi_e}q \tag{25}$$

which is the result given in Griffiths's book for that case. Further, if we let $\chi_e\to\infty$, we get $Q=-q$, which is the result for a point charge next to a conducting plane. Also, if we let $\chi_e'\to\infty$, we get $Q=0$, since in that case we've embedded q inside a conductor, which shields it completely, so there is no induced charge at the boundary.

Getting back to the method of images, we have to find the potential in the two regions. For $z>0$, we have a shielded charge with effective charge q/ε_r' at $z=d$ and we want to replace the surface charge with a point image in the region $z<0$. What should be the amount of this image charge? In the case of the conducting plane, the image charge was the same amount as the total surface charge, and at location $z=-d$, so we can try that. In this case, we get

$$V_{above}=\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\varepsilon_r'\sqrt{r^2+(z-d)^2}}+\frac{Q}{\sqrt{r^2+(z+d)^2}}\right] \tag{26}$$

For $z<0$, we still see the original charge at $z=+d$, and we can try replacing the surface charge by a point charge the same distance beyond the plane as we did when calculating the image for the $z>0$ region. That is, we place an image charge of size Q at $z=+d$ (so it coincides with the original point charge). In this case, the potential is

$$V_{below}=\frac{1}{4\pi\varepsilon_0}\frac{q/\varepsilon_r+Q}{\sqrt{r^2+(z-d)^2}} \tag{27}$$

We can check that this potential satisfies the required boundary conditions. First, it must be continuous at $z=0$, which it obviously is:

$$
\begin{aligned}
V_{above}(z &= 0) = \frac{1}{4\pi\varepsilon_0} \frac{q/\varepsilon_r' + Q}{\sqrt{r^2 + z^2}} && (28) \\
&= V_{below}(z = 0) && (29)
\end{aligned}
$$

Second, we should be able to derive the discontinuity in the field above. We have

$$
\left.\frac{\partial V_{above}}{\partial z}\right|_{z=0} = \frac{d}{4\pi\varepsilon_0 \left(r^2 + d^2\right)^{3/2}} \left[\frac{q}{\varepsilon_r'} - Q\right] \tag{30}
$$

$$
\left.\frac{\partial V_{below}}{\partial z}\right|_{z=0} = \frac{d}{4\pi\varepsilon_0 \left(r^2 + d^2\right)^{3/2}} \left[\frac{q}{\varepsilon_r'} + Q\right] \tag{31}
$$

Since $E_z = -\partial V / \partial z$ we get

$$
\begin{aligned}
E_{z(above)} - E_{z(below)} &= \frac{Qd}{2\pi\varepsilon_0 \left(r^2 + d^2\right)^{3/2}} && (32) \\
&= \frac{\left(\chi_e' - \chi_e\right) qd}{2\pi\varepsilon_0 \left(r^2 + d^2\right)^{3/2} \left(1 + \chi_e'\right)\left(2 + \chi_e + \chi_e'\right)} && (33) \\
&= \frac{\sigma_b + \sigma_b'}{\varepsilon_0} && (34)
\end{aligned}
$$

So this checks out as well.