ENERGY OF CONDUCTING SPHERE IN A DIELECTRIC SHELL

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Section 4.4.3 & Problem 4.26.

We've seen that the energy in a system containing dielectrics can be written as

$$W=\frac{1}{2}\int(\mathbf{D}\cdot\mathbf{E})\,d^3\mathbf{r} \tag{1}$$

This is the energy required to place the free charges, and includes the energy needed to polarize the dielectric.

As a simple example of this formula, suppose we have a spherical conductor of radius a that has a free charge Q on it, and we surround this conductor with a spherical shell of linear dielectric that extends from $r=a$ to $r=b$. The free charge on the conductor will polarize the dielectric, resulting in surface charges on the inner and outer surfaces of the dielectric.

We can find the displacement $\mathbf{D}$ from its relation to the free charge. That is,

$$\int_A \mathbf{D}\cdot d\mathbf{a}=Q_f \tag{2}$$

where Q_f is the free charge (excluding the bound charge) that is contained within the surface of integration.

In this case, $Q_f=Q$ and because the system has spherical symmetry, we can take the surface of integration to be a sphere. Since the enclosed free charge is Q for any surface with $r>a$, we get in this region:

$$\begin{aligned} 4\pi r^2 D &= Q && (3)\\ \mathbf{D} &= \frac{Q}{4\pi r^2}\hat{\mathbf{r}} && (4)\end{aligned}$$

For a linear dielectric $\mathbf{D}=\varepsilon\mathbf{E}=\varepsilon_0(1+\chi_e)\mathbf{E}$. We therefore have for the electric field:

$$\mathbf{E}=\begin{cases}\frac{Q}{4\pi\varepsilon_0(1+\chi_e)r^2}\hat{\mathbf{r}} & a<r<b\\ \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} & r>b\end{cases} \tag{5}$$

The energy is then

$$W=\frac{1}{2}\int(\mathbf{D}\cdot\mathbf{E})\,d^3\mathbf{r} \tag{6}$$

$$=\frac{4\pi Q^2}{2(4\pi)^2\varepsilon_0}\left[\int_a^b\frac{r^2dr}{(1+\chi_e)r^4}+\int_b^\infty\frac{r^2dr}{r^4}\right] \tag{7}$$

$$=\frac{Q^2}{8\pi\varepsilon_0}\left[\frac{1}{1+\chi_e}\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{1}{b}\right] \tag{8}$$

$$=\frac{Q^2}{8\pi\varepsilon_0(1+\chi_e)}\left(\frac{1}{a}+\frac{\chi_e}{b}\right) \tag{9}$$

If we remove the dielectric, this is the same as setting $\chi_e=0$, so the energy stored in the field of a charged conducting sphere is

$$W=\frac{Q^2}{8\pi\varepsilon_0 a} \tag{10}$$