

ENERGY OF CONDUCTING SPHERE IN A DIELECTRIC SHELL

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Section 4.4.3 & Problem 4.26.

We've seen that the energy in a system containing dielectrics can be written as

$$(1) \quad W = \frac{1}{2} \int (\mathbf{D} \cdot \mathbf{E}) d^3\mathbf{r}$$

This is the energy required to place the free charges, and includes the energy needed to polarize the dielectric.

As a simple example of this formula, suppose we have a spherical conductor of radius a that has a free charge Q on it, and we surround this conductor with a spherical shell of linear dielectric that extends from $r = a$ to $r = b$. The free charge on the conductor will polarize the dielectric, resulting in surface charges on the inner and outer surfaces of the dielectric.

We can find the displacement \mathbf{D} from its relation to the free charge. That is,

$$(2) \quad \int_A \mathbf{D} \cdot d\mathbf{a} = Q_f$$

where Q_f is the free charge (excluding the bound charge) that is contained within the surface of integration.

In this case, $Q_f = Q$ and because the system has spherical symmetry, we can take the surface of integration to be a sphere. Since the enclosed free charge is Q for any surface with $r > a$, we get in this region:

$$(3) \quad 4\pi r^2 D = Q$$

$$(4) \quad \mathbf{D} = \frac{Q}{4\pi r^2} \hat{\mathbf{r}}$$

For a linear dielectric $\mathbf{D} = \epsilon\mathbf{E} = \epsilon_0(1 + \chi_e)\mathbf{E}$. We therefore have for the electric field:

$$(5) \quad \mathbf{E} = \begin{cases} \frac{Q}{4\pi\epsilon_0(1+\chi_e)r^2}\hat{\mathbf{r}} & a < r < b \\ \frac{Q}{4\pi\epsilon_0r^2}\hat{\mathbf{r}} & r > b \end{cases}$$

The energy is then

$$(6) \quad W = \frac{1}{2} \int (\mathbf{D} \cdot \mathbf{E}) d^3\mathbf{r}$$

$$(7) \quad = \frac{4\pi Q^2}{2(4\pi)^2 \epsilon_0} \left[\int_a^b \frac{r^2 dr}{(1+\chi_e)r^4} + \int_b^\infty \frac{r^2 dr}{r^4} \right]$$

$$(8) \quad = \frac{Q^2}{8\pi\epsilon_0} \left[\frac{1}{1+\chi_e} \left(\frac{1}{a} - \frac{1}{b} \right) + \frac{1}{b} \right]$$

$$(9) \quad = \frac{Q^2}{8\pi\epsilon_0(1+\chi_e)} \left(\frac{1}{a} + \frac{\chi_e}{b} \right)$$

If we remove the dielectric, this is the same as setting $\chi_e = 0$, so the energy stored in the field of a charged conducting sphere is

$$(10) \quad W = \frac{Q^2}{8\pi\epsilon_0 a}$$