

## ENERGY OF CONDUCTING SPHERE IN A DIELECTRIC SHELL

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Section 4.4.3 & Problem 4.26.

We've seen that the energy in a system containing dielectrics can be written as

$$W = \frac{1}{2} \int (\mathbf{D} \cdot \mathbf{E}) d^3\mathbf{r} \quad (1)$$

This is the energy required to place the free charges, and includes the energy needed to polarize the dielectric.

As a simple example of this formula, suppose we have a spherical conductor of radius  $a$  that has a free charge  $Q$  on it, and we surround this conductor with a spherical shell of linear dielectric that extends from  $r = a$  to  $r = b$ . The free charge on the conductor will polarize the dielectric, resulting in surface charges on the inner and outer surfaces of the dielectric.

We can find the displacement  $\mathbf{D}$  from its relation to the free charge. That is,

$$\int_A \mathbf{D} \cdot d\mathbf{a} = Q_f \quad (2)$$

where  $Q_f$  is the free charge (excluding the bound charge) that is contained within the surface of integration.

In this case,  $Q_f = Q$  and because the system has spherical symmetry, we can take the surface of integration to be a sphere. Since the enclosed free charge is  $Q$  for any surface with  $r > a$ , we get in this region:

$$4\pi r^2 D = Q \quad (3)$$

$$\mathbf{D} = \frac{Q}{4\pi r^2} \hat{\mathbf{r}} \quad (4)$$

For a linear dielectric  $\mathbf{D} = \epsilon\mathbf{E} = \epsilon_0(1 + \chi_e)\mathbf{E}$ . We therefore have for the electric field:

$$\mathbf{E} = \begin{cases} \frac{Q}{4\pi\epsilon_0(1+\chi_e)r^2}\hat{\mathbf{r}} & a < r < b \\ \frac{Q}{4\pi\epsilon_0r^2}\hat{\mathbf{r}} & r > b \end{cases} \quad (5)$$

The energy is then

$$W = \frac{1}{2} \int (\mathbf{D} \cdot \mathbf{E}) d^3\mathbf{r} \quad (6)$$

$$= \frac{4\pi Q^2}{2(4\pi)^2\epsilon_0} \left[ \int_a^b \frac{r^2 dr}{(1+\chi_e)r^4} + \int_b^\infty \frac{r^2 dr}{r^4} \right] \quad (7)$$

$$= \frac{Q^2}{8\pi\epsilon_0} \left[ \frac{1}{1+\chi_e} \left( \frac{1}{a} - \frac{1}{b} \right) + \frac{1}{b} \right] \quad (8)$$

$$= \frac{Q^2}{8\pi\epsilon_0(1+\chi_e)} \left( \frac{1}{a} + \frac{\chi_e}{b} \right) \quad (9)$$

If we remove the dielectric, this is the same as setting  $\chi_e = 0$ , so the energy stored in the field of a charged conducting sphere is

$$W = \frac{Q^2}{8\pi\epsilon_0 a} \quad (10)$$