ENERGY IN A DIELECTRIC

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Section 4.4.3 & Problem 4.27.

We now have a look at the analogous formula when we have free charges embedded in a dielectric. To find the work required to assemble *only* the free charges (excluding the bound charges), we start with the ideas developed originally for finding the energy required to add a point charge to an existing configuration of charges. Assuming that the potential is zero at infinity and we bring in the charge q from there to point $\mathbf{r}$, the energy required to do this is $W = qV(\mathbf{r})$.

In the continuous case, if we bring in a small amount $\Delta\rho_f$ of free charge, then the energy required to do this is

$$\Delta W = \int \left[\Delta\rho_f(\mathbf{r})\right] V(\mathbf{r})\, d^3\mathbf{r} \tag{1}$$

(Note that it is incorrect to start with the formula $W = \frac{1}{2}\int \rho(\mathbf{r}) V(\mathbf{r})\, d^3\mathbf{r}$, since that formula was derived for a situation where we considered all the charges (not just the free charges) already in place, so that the potential in the formula is from the final distribution of charges. In our case, we're essentially retracing the derivation used originally for the case of adding a point charge to an existing distribution and building up the formula from there.)

Since $\nabla\cdot\mathbf{D} = \rho_f$ (see here), we can write this as

$$\Delta W = \int \nabla\cdot[\Delta\mathbf{D}]\, V(\mathbf{r})\, d^3\mathbf{r} \tag{2}$$

Using the vector calculus identity

$$\nabla\cdot(\mathbf{A}V) = V\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla V \tag{3}$$

and the relation between field and potential $\mathbf{E} = -\nabla V$, we get

$$\Delta W = \int \nabla\cdot[V\Delta\mathbf{D}]\, d^3\mathbf{r} + \int \Delta\mathbf{D}\cdot\mathbf{E}\, d^3\mathbf{r} \tag{4}$$

Using the divergence theorem, we can convert the first integral into a surface integral and let the surface go to infinity, making the usual assumption that the quantity $V\Delta\mathbf{D}$ goes to zero faster than $1/r^2$, so that the surface integral also goes to zero. We therefore end up with

$$\Delta W = \int \Delta\mathbf{D}\cdot\mathbf{E}d^3\mathbf{r} \tag{5}$$

Now for a linear dielectric, $\mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E}$, so we can write, for incremental changes

$$\Delta(\mathbf{D}\cdot\mathbf{E}) = \varepsilon_0\varepsilon_r\Delta(\mathbf{E}\cdot\mathbf{E}) \tag{6}$$

$$= \varepsilon_0\varepsilon_r\Delta E^2 \tag{7}$$

$$= 2\varepsilon_0\varepsilon_r E\Delta E \tag{8}$$

$$= 2\mathbf{E}\cdot\Delta\mathbf{D} \tag{9}$$

So if we pull the physicist's trick of interchanging differentials and integrals, we can write

$$\Delta W = \frac{1}{2}\Delta\int(\mathbf{D}\cdot\mathbf{E})\,d^3\mathbf{r} \tag{10}$$

So in general, the work done to assemble free charge in a dielectric is

$$W = \frac{1}{2}\int(\mathbf{D}\cdot\mathbf{E})\,d^3\mathbf{r} \tag{11}$$

Earlier we saw that the energy of a static charge distribution could be written in terms of the electric field:

$$W = \frac{\varepsilon_0}{2}\int E^2 d^3\mathbf{r} \tag{12}$$

If we compare these two formulas they appear to be incompatible, since if $\mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E}$, the first integral becomes

$$W = \frac{\varepsilon_0\varepsilon_r}{2}\int E^2 d^3\mathbf{r} \tag{13}$$

so that there's a factor of ε_r in the first integral that is absent from the second.

The resolution of this dilemma is that we derived the dielectric formula for *free charge* only. In the absence of a dielectric, all charge is free charge, and $\varepsilon_r = 1$ so the two formulas become the same. However, when we have dielectric present, bringing in free charge will cause a polarization of the dielectric, which requires energy, since we are essentially pulling apart the positive and negative bound charges in the dielectric's atoms. Thus the energy required to place a free charge next to a dielectric is greater (by the factor ε_r) than that required to place the same charge without the dielectric being there.

As an example of the difference between the two formulas, consider again the problem of the uniformly polarized sphere with polarization $\mathbf{P}$. We found that the field inside the sphere was uniform:

$$\mathbf{E}_{r<R} = -\frac{1}{3\varepsilon_0}\mathbf{P} \tag{14}$$

Outside the sphere, the potential is

$$V_{r>R} = \frac{R^3}{3\varepsilon_0 r^2} P\cos\theta \tag{15}$$

This is the same as the potential of a pure dipole with dipole moment

$$\mathbf{p} = \frac{4}{3}\pi R^3 \mathbf{P} \tag{16}$$

Thus the field outside the sphere is the same as that of a pure dipole, so we get

$$\mathbf{E}_{r>R} = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right) \tag{17}$$

Outside the sphere, the energy is the same no matter which formula we use since there is no dielectric here. We therefore have

$$\begin{aligned}
W_{r>R} &= \frac{1}{2}\varepsilon_0 \int E^2 d^3\mathbf{r} && (18)\\
&= \frac{p^2\varepsilon_0}{2\left(4\pi\varepsilon_0\right)^2} 2\pi \int_R^\infty \int_0^\pi \frac{r^2\sin\theta}{r^6}\left(4\cos^2\theta + \sin^2\theta\right) d\theta dr && (19)\\
&= \frac{p^2}{12\pi\varepsilon_0 R^3} && (20)\\
&= \frac{4\pi P^2 R^3}{27\varepsilon_0} && (21)
\end{aligned}$$

Inside the sphere, since the field is constant, we have, using the total charge formula

$$W_{r<R} = \frac{\varepsilon_0}{2}\int E^2 d^3\mathbf{r} \tag{22}$$

$$= \frac{2\pi P^2 R^3}{27\varepsilon_0} \tag{23}$$

The total energy calculated this way is thus

$$W_{tot} = \frac{4\pi P^2 R^3}{27\varepsilon_0} + \frac{2\pi P^2 R^3}{27\varepsilon_0} \tag{24}$$

$$= \frac{2\pi P^2 R^3}{9\varepsilon_0} \tag{25}$$

Calculated using the free charge formula, we use the definition of displacement as $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = -\mathbf{P}/3 + \mathbf{P} = 2\mathbf{P}/3$. Then

$$W_{r<R} = \frac{1}{2}\int (\mathbf{D}\cdot\mathbf{E})\, d^3\mathbf{r} \tag{26}$$

$$= -\frac{4\pi P^2 R^3}{27\varepsilon_0} \tag{27}$$

The total energy in this case thus comes out to $W_{tot} = 0$, which is because there is no free charge in the problem. However, the derivation above also relied on the dielectric being *linear*, which isn't the case in this problem since we have a frozen in polarization with no external electric field. Thus the second result doesn't really mean much anyway.

PINGBACKS

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